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THE SOLUTION OF BOUNDARY VALUE PROBLEMS OF VARIOUS TYPES WITH CONSIDERATION OF VOLUME FORCES FOR ANISOTROPIC BODIES OF REVOLUTION

D.A. Ivanychev[1] lsivdmal@mail.ru
E.Yu. Levina[2] levina@bmstu.ru

[1] Lipetsk State Technical University, Lipetsk, Russian Federation
[2] Bauman Moscow State Technical University, Moscow, Russian Federation

Abstract

In this work, we studied the axisymmetric elastic equilibrium of transversely isotropic bodies of revolution, which are simultaneously under the influence of surface and volume forces. The construction of the stress-strain state is carried out by means of the boundary state method. The method is based on the concepts of internal and boundary states conjugated by an isomorphism. The bases of state spaces are formed, orthonormalized, and the desired state is expanded in a series of elements of the orthonormal basis. The Fourier coefficients, which are quadratures, are calculated. In this work, we propose a method for forming bases of spaces of internal and boundary states, assigning a scalar product and forming a system of equations that allows one to determine the elastic state of anisotropic bodies. The peculiarity of the solution is that the obtained stresses simultaneously satisfy the conditions both on the boundary of the body and inside the region (volume forces), and they are not a simple superposition of elastic fields. Methods are presented for solving the first and second main problems of mechanics, the contact problem without friction and the main mixed problem of the elasticity theory for transversely isotropic finite solids of revolution that are simultaneously under the influence of volume forces. The given forces are distributed axisymmetrically with respect to the geometric axis of rotation. The solution of the first main problem for a non-canonical body of revolution is given, an analysis of accuracy is carried out and a graphic illustration of the result is given

Keywords

The method of boundary states, boundary value problems of mechanics, anisotropy, bulk forces, state space, elastic equilibrium

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Introduction. Machine parts and mechanisms made of modern anisotropic materials are subjected to the simultaneous action of surface and volumetric forces. Kinematic conditions are imposed on their boundary. Determination of the stress-strain state (SSS) from the totality of such effects and taking into account the complexity of the structure of the material is an urgent scientific task.

Volume forces are considered in various works on mechanics. So, in [1] problems of the elasticity theory with given volumetric and surface forces in functional energy spaces of stress and strain tensors are considered; specific problems are solved by the method of orthogonal projections. Exact analytical solutions of equilibrium problems for thick-walled transversely isotropic compound spheres with a rigidly fixed boundary and under the action of mass forces and internal pressure were obtained in [2]. Using the variational Lagrange equation, in [3] the condition of equivalence of surface and volume forces was obtained for displacements. The displacement field for an elastic body bounded by concentric spheres and under the action of axisymmetric unsteady volumetric forces was constructed in [4].

Contact problems of the elasticity theory have been sufficiently studied; more specific aspects of contact problems are currently being investigated. For example, in [5] the authors studied a spatial contact problem with an unknown contact region for a transversely isotropic elastic half-space. To solve the problem, they used Galanov's numerical method, which allows one to simultaneously determine the contact area and the pressure in this area. The calculation of the contact of cylinders with finite length taking into account the action of the edge effect, its influence on the contact area and stress is given in [6]. Using the Fourier transform, the mixed Dirichlet — Neumann boundary value problem for the Poisson equation in a domain bounded by two parallel hyperplanes was solved in [7]. The solution is written in terms of the constructed Green's function of the Laplace operator.

Mixed type boundary-value problems of the elasticity theory were considered less often than problems with the same type of boundary conditions, but their study was carried out in application to various areas of mechanics. Mathematical and numerical analysis of asymptotic solutions of three-dimensional static problems of elasticity theory with mixed boundary conditions was performed in [8]. An algorithm for the numerical solution of a mixed problem of the elasticity theory for a body that has contact interaction with an elastic half-space is implemented in [9]. An analytical solution of the plane mixed problem of the elasticity theory for a doubly connected domain was constructed in [10].

The boundary state method is used to solve problems for simply connected and multiply connected domains of anisotropic bodies. For example, in [11], a technique was demonstrated for the formation of a basis of internal states for a plane multiply connected anisotropic region. The boundary state method in [12] was applied to solve the problems of torsion of infinite anisotropic solid cylinders. Thermoelastic equilibrium of transtropic bodies of revolution under the action of mass forces was studied in [13].

Problem statement. A transversely isotropic body is considered, bounded by one or several coaxial surfaces of revolution (Fig. 1). The anisotropy axis coincides with the geometric z axis. A cylindrical coordinate system z, r was used. The body is acted upon by volume forces $\mathbf{X}^* = \{R^*, Z^*\}$ and, in contrast to the type of problem, forces $\mathbf{p}^* = \{p_r^*, p_z^*\}$ (the first main task), displacements of the boundary points $\mathbf{u}^* = \{u^*, w^*\}$ (the second main task), or a combination of both on the boundary sections (the main mixed task) are set on the body surface. The task is to determine the stress-strain state arising in the body under the influence of these factors.

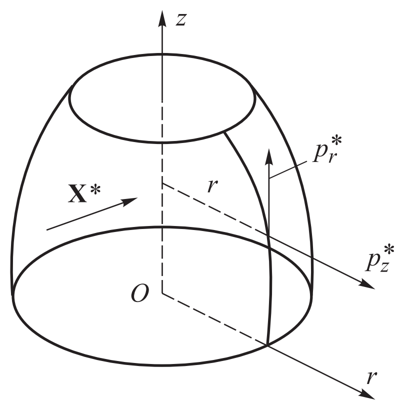


Fig. 1. Transversely isotropic body of revolution

Materials and methods. To solve the problem, the boundary state method (BSM) was used [14]. The method is based on the spaces of internal Ξ and boundary Γ states:

$$\Xi = \{\xi_k\}_N; \ \Gamma = \{\gamma_k\}_N,$$

where ξ_k, γ_k are the basic elements of these spaces, $k = 1, 2, \ldots, N$; N is the number of basic elements.

The basic element ξ_k in space Ξ is a certain internal state, which includes the displacement vector $\mathbf{u}^k = \{u^k, w^k\}$, the strain tensor $\hat{\varepsilon}^k = \varepsilon_{ij}^k$ and the stress tensor $\hat{\sigma}^k = \sigma_{ij}^k$:

$$\xi_k = \{\mathbf{u}^k, \hat{\varepsilon}^k, \hat{\sigma}^k\} \tag{1}$$

(on the right side of the expression, the index k is placed at the top).

In the traditional BSM, the basic element γ_k in space Γ is a certain boundary state corresponding to ξ_k:

$$\gamma_k = \{\mathbf{u}_v^k, \mathbf{p}^k\},$$

where $\mathbf{u}_v^k = \{u_v^k, w_v^k\}$ is the movement vector of the border points; $\mathbf{p}^k = \{p_r^k, p_z^k\}$ is the force vector at the border.

Here the vector of volumetric forces $\mathbf{X}^k = \{R^k, Z^k\}$ is included in the boundary state, which is conditional because volumetric forces do not relate to the boundary of the body:

$$\gamma_k = \{\mathbf{u}_v^k, \mathbf{p}^k, \mathbf{X}^k\}. \tag{2}$$

After assigning the bases of the state spaces, it is necessary to ensure their isomorphism; for this, the Clapeyron equation [15] was used:

$$\int_V \mathbf{X}\,\mathbf{u}\,dV + \int_S \mathbf{p}\,\mathbf{u}_v\,dS - \int_V \hat{\sigma}\,\hat{\varepsilon}\,dV = 0. \tag{3}$$

According to (3), the scalar product in space Ξ can be determined using the internal energy of elastic deformation (for a body occupying region V, for example, for states 1 and 2):

$$(\xi_1, \xi_2) = \int_V \hat{\varepsilon}^1 \hat{\sigma}^2 dV, \tag{4}$$

and in space Γ it can be determined using the work of external forces on the surface of the body S and the work of volumetric forces on the movements of the internal points of the body:

$$(\gamma_1, \gamma_2) = \int_S \mathbf{p}^1 \mathbf{u}_v^2 dS + \int_V \mathbf{X}^1 \mathbf{u}^2 dV. \tag{5}$$

Moreover, due to the Betti identity and the Clapeyron relation $(\xi_1, \xi_2) = (\xi_2, \xi_1)$; $(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1)$. Each element $\xi_k \in \Xi$ is in one-to-one correspondence with a single element $\gamma_k \in \Gamma$. This allows us to reduce the search for an internal state to the construction of a boundary state isomorphic to it.

After constructing the basis of internal states, the technique of which is described below for each problem, it is necessary to orthonormalize it. It is carried out according to the developed recursive-matrix Gram — Schmidt algorithm [16], where relation (4) is taken as the cross scalar products.

The orthonormalized basis Γ is reduced from the orthonormalized basis Ξ by passing to the limit to the body boundary for the displacement vector $\mathbf{u}_v^k = \mathbf{u}^k \big|_S$, calculating the forces at the boundary using the relation $p_l^k = \sigma_{lp}^k n_p$ (in the cylindrical coordinate system $l, p = r, z$; n_p is the component of the normal to the boundary) and volume forces from the equilibrium equations [17].

However, it is possible to carry out the basis orthonormalization of the space Γ separately, operating with the basis elements of this space and the scalar product (5).

Further, the course of the solution depends on the task.

The first main task. Volumetric forces $\mathbf{X}^* = \{R^*, Z^*\}$ inside the region and forces $\mathbf{p}^* = \{p_r^*, p_z^*\}$ on the entire boundary are given.

The desired elastic state $\xi = \{\mathbf{u}, \hat{\varepsilon}, \hat{\sigma}\}$ and its trace on the boundary $\gamma = \{\mathbf{u}_\nu, \mathbf{p}, \mathbf{X}\}$ are the Fourier series

$$\xi = \sum_{k=1}^{N} c_k \xi_k; \ \gamma = \sum_{k=1}^{N} c_k \gamma_k \tag{6}$$

or if expanded for internal state

$$\mathbf{u} = \sum_{k=1}^{\infty} c_k \mathbf{u}^k; \ \hat{\varepsilon} = \sum_{k=1}^{\infty} c_k \hat{\varepsilon}^k; \ \hat{\sigma} = \sum_{k=1}^{\infty} c_k \hat{\sigma}^k.$$

The series coefficients are the same and are calculated by the formula

$$c_k = \int_V \mathbf{X}^* \mathbf{u}^k dV + \int_S \mathbf{p}^* \mathbf{u}_\nu^k dS.$$

Here $\mathbf{u}$, $\mathbf{u}_\nu^k$ are the displacement vectors in the k-th basic element (1) of the internal states space and in the k-th element (2) of the boundary states space.

The second main task. Volumetric forces $\mathbf{X}^* = \{R^*, Z^*\}$ and displacements of boundary points $\mathbf{u}_\nu^* \in \{u_\nu^*, w_\nu^*\}$ are set.

Orthonormalization of the state spaces bases allows us to write the following (i, j are some numbers of basic elements, $i, j = 1, 2, 3, \ldots, N$):

$$\int_V \mathbf{X}^i \mathbf{u}^j dV + \int_S \mathbf{p}^j \mathbf{u}_\nu^i dS + \int_V \mathbf{X}^j \mathbf{u}^i dV + \int_S \mathbf{p}^i \mathbf{u}_\nu^j dS = 2\delta_{ij}, \tag{7}$$

where δ_{ij} is the Kronecker delta. Wherein

$$\int_V \mathbf{X}^i \mathbf{u}^j dV + \int_S \mathbf{p}^j \mathbf{u}_\nu^i dS = -\left[\int_V \mathbf{X}^j \mathbf{u}^i dV + \int_S \mathbf{p}^j \mathbf{u}_\nu^i dS \right], \ i \neq j.$$

Replacing the basic characteristics of the elastic field $\mathbf{X}^j$, $\mathbf{u}_\nu^j$ in (7) with the given ones $\mathbf{X}^*$, $\mathbf{u}_\nu^*$ and performing an enumeration over the index j, we form the matrix of coefficients A and B:

$$\beta_{ij} = \int_V \mathbf{X}^i \mathbf{u}^j dV + \int_S \mathbf{p}^j \mathbf{u}_\nu^i dS; \ \alpha_j = \int_V \mathbf{X} \mathbf{u}^j dV + \int_S \mathbf{p}^j \mathbf{u}_\nu dS;$$
$$\mathrm{A} = [\alpha_j]_N; \ \mathrm{B} = [\beta_{ij}]_{N \times N}; \ \beta_{ij} = -\beta_{ji}, \ i \neq j. \tag{8}$$

Here it is convenient to introduce a matrix-column of Fourier coefficients $C = [c_k]_N$, which is determined by the relation:

$$C = B^{-1}A, \tag{9}$$

where N is the number of basis elements used. The final solution has the form (6).

Mixed task. The volumetric forces $\mathbf{X}^*$, forces $\mathbf{p}^*$ on the surface section S_p and the displacement of the boundary points $\mathbf{u}_\nu^*$ on the surface section S_u, $S = S_p + S_u$ are given.

Representing the terms from (7) in the form

$$\int_S \mathbf{p}^i\,\mathbf{u}_\nu^j dS = \int_{Sp} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_p + \int_{Su} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_u;\ \int_S \mathbf{p}^j \mathbf{u}_\nu^i dS = \int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p + \int_{Su} \mathbf{p}^j \mathbf{u}_\nu^i dS_u$$

and substituting them into (7), we obtain

$$2\int_V \mathbf{X}^i \mathbf{u}^j dV + \int_{Sp} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_p + \int_{Su} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_u + \int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p + \int_{Su} \mathbf{p}^j \mathbf{u}_\nu^i dS_u +$$

$$+2\int_V \mathbf{X}^j \mathbf{u}^i dV + \int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p + \int_{Su} \mathbf{p}^j \mathbf{u}_\nu^i dS_u + \int_{Sp} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_p + \int_{Su} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_u = 2\delta_{ij}.$$

Let us group the terms and denote:

$$\beta_{ij} = 2\int_V \mathbf{X}^i \mathbf{u}^j dV + 2\int_{Sp} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_p + 2\int_{Su} \mathbf{p}^j \mathbf{u}_\nu^i dS_u;$$

$$\lambda_{ij} = 2\int_V \mathbf{X}^j \mathbf{u}^i dV + 2\int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p + 2\int_{Su} \mathbf{p}^i\,\mathbf{u}_\nu^j dS_u;\ \beta_{ij} + \lambda_{ij} = 2\delta_{ij}.$$

Let us transform λ_{ij} in the following way: we replace the basic characteristics of the elastic field $\mathbf{X}^j$, $\mathbf{p}^j$, $\mathbf{u}_\nu^j$ with the given ones, and we will enumerate the coefficients by the index *j*; then

$$\alpha_j = 2\int_V \mathbf{X}^*\,\mathbf{u}^j dV + 2\int_{Sp} \mathbf{p}^* \mathbf{u}_\nu^j dS_p + 2\int_{Su} \mathbf{p}^j \mathbf{u}^* dS_u.$$

Next, we form matrices A and B (8), calculate column C using dependence (9), and determine the desired characteristics by (6).

Let us now consider a mixed boundary value problem for which normal displacements and shear stresses (in terms of BSM these are forces) are specified on one of the boundaries of the body. This setting simulates the contact interaction of the body under consideration with another body or half-space.

Here the volumetric forces $\mathbf{X}^*$ are set, as well as the boundary conditions on the contact surface $\{p_r^*, u_z^*\}|S_u$ and the forces on the rest of the body surface $\mathbf{p}^* = \{p_r^*, p_z^*\}|S_p$.

Representing the terms from (7) in the form

$$\int_S \mathbf{p}^i \mathbf{u}_\nu^j dS = \int_{Su} \mathbf{p}^i \mathbf{u}_\nu^j dS_u + \int_{Sp} \mathbf{p}^i \mathbf{u}_\nu^j dS_p = \int_{Su} p_r^i u_r^j dS_u + \int_{Su} p_z^i u_z^j dS_u + \int_{Sp} \mathbf{p}^i \mathbf{u}_\nu^j dS_p;$$

$$\int_S \mathbf{p}^j \mathbf{u}_\nu^i dS = \int_{Su} p_r^j u_r^i dS_u + \int_{Su} p_z^j u_z^i dS_u + \int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p,$$

and substituting them into (7), we obtain

$$2\int_V \mathbf{X}^i \mathbf{u}^j dV + 2\int_{Su} p_r^i u_r^j dS_u + 2\int_{Su} p_z^i u_z^j dS_u + 2\int_{Sp} \mathbf{p}^i \mathbf{u}_\nu^j dS_p + 2\int_V \mathbf{X}^j \mathbf{u}^i dV +$$
$$+ 2\int_{Su} p_r^j u_r^i dS_u + 2\int_{Su} p_z^j u_z^i dS_u + 2\int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p = 2\delta_{ij}.$$

Let us group the terms and denote

$$\beta_{ij} = 2\int_V \mathbf{X}^i \mathbf{u}^j dV + 2\int_{Su} p_r^i u_r^j dS_u + 2\int_{Su} p_z^j u_z^i dS_u + 2\int_{Sp} \mathbf{p}^i \mathbf{u}_\nu^j dS_p;$$

$$\lambda_{ij} = 2\int_V \mathbf{X}^j \mathbf{u}^i dV + 2\int_{Su} p_r^j u_r^i dS_u + 2\int_{Su} p_z^i u_z^j dS_u + 2\int_{Sp} \mathbf{p}^j \mathbf{u}_\nu^i dS_p,$$

It is easy to see that $\beta_{ij} + \lambda_{ij} = 2\delta_{ij}$.

Let us transform λ_{ij} as follows: we replace the basic characteristics of the elastic field $\mathbf{X}^j$, p_r^j, u_z^j with the given ones, we will carry out the enumeration by the index *j*, thus forming the matrix of coefficients A (8), for which

$$\alpha_j = 2\int_V \mathbf{X}^* \mathbf{u}^j dV + 2\int_{Su} p_r^* u_r^j dS_u + 2\int_{Su} p_z^j u_z^* dS_u + 2\int_{Sp} \mathbf{p}^* \mathbf{u}_\nu^j dS_p.$$

Further, the solution is constructed using dependencies (9) and (6).

In all problems, the testing of the Fourier coefficients is carried out by substituting one of the basic elements with the corresponding boundary conditions as a given one, while the conditions $c_n = 1$ (*n* is the number of the tested element) must be satisfied, the rest of the Fourier coefficients must be equal to zero.

Formation of the internal states basis. The main task in the boundary states method is the formation of the internal states basis, which can be conditionally divided into two: 1) the basis in the elastostatics problem (space Ξ^S); 2) the basis in the problem of volumetric forces (space Ξ^X).

The basis formation technique for the elastostatic problem is described in detail in [18]. Based on the method of integral superposition, they established the dependence of the spatial stressed and deformed states of an elastic transversely isotropic body on certain auxiliary two-dimensional states, the components of which depend on two coordinates [17]. Plane deformation is used as such states:

$$u_r^{pl} = \mathrm{Re}[iq_1\phi_1(\varsigma_1) + iq_2\phi_2(\varsigma_2)];\; u_z^{pl} = \mathrm{Re}[p_1\phi_1(\varsigma_1) + p_2\phi_2(\varsigma_2)], \qquad (10)$$

where q_1, p_1 are the constants determined by the elastic constants of the transtropic material; $\varsigma_j = z/\gamma_j + iy$, γ_j are complex roots of the characteristic equation, functions $\phi_j(\varsigma_j)$ are analytical in their variables.

The transition from a plane state to an axisymmetric spatial state in cylindrical coordinates *z, r* is carried out according to the dependences [17]:

$$u = \frac{1}{\pi}\int_{-r}^{r} \frac{u_y^{pl}}{r\sqrt{r^2 - y^2}}dy;\; w = \frac{1}{\pi}\int_{-r}^{r} \frac{u_z^{pl}}{r\sqrt{r^2 - y^2}}dy;\; v = 0. \qquad (11)$$

Giving the values successively to two analytical functions in (10):

$$\begin{pmatrix}\phi_1(\varsigma_1)\\ \phi_2(\varsigma_2)\end{pmatrix} \in \left\{\begin{pmatrix}\varsigma_1^n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \varsigma_2^n\end{pmatrix}, \begin{pmatrix}i\varsigma_1^n\\ 0\end{pmatrix}, \begin{pmatrix}0\\ i\varsigma_2^n\end{pmatrix}, \ldots\right\},\; n = 1, 2, 3, \ldots$$

All elastic characteristics of the plane auxiliary state are determined, and then a transition to a three-dimensional state follows according to dependencies (11), forming a finite-dimensional basis in the problem of elastostatics: $\Xi^S = \left\{\xi_k^S\right\}$.

To construct plane auxiliary states from the action of volumetric forces, a fundamental system of polynomials $y^\alpha z^\beta$, was applied, which can be placed in any position of the displacement vector $\mathbf{u}^{pl}$, forming a certain admissible elastic field [19]:

$$\mathbf{u}^{pl} = \{\{y^\alpha z^\beta, 0\}, \{0, y^\alpha z^\beta\}\}.$$

Enumeration of all possible options within the limits $\alpha + \beta \le n$, $n =$ 1, 2, 3,…, allows you to get a set of plane states. Further, according to (11), the components of the displacement vector of the spatial axisymmetric state are determined and the corresponding deformations tensors, stresses tensors and volumetric forces are determined, forming a finite-dimensional basis in the problem of volumetric forces: $\Xi^X = \left\{ \xi_k^X \right\}$.

In the first main task, the following basis is used:

$$\Xi = \left\{ \xi_k^X \right\}, \quad k = 1, 2, 3, \dots \tag{12}$$

It is built only on the elements of states from the action of volumetric forces.

In all other tasks, the combined basis is used:

$$\Xi = \left\{ \xi_k^S, \xi_{k+1}^X \right\}, \quad k = 1, 3, 5, 7, 9, \dots,$$

otherwise, the convergence of the solution will not be ensured.

Problem solution. Let us consider the first main problem for a body of revolution of a non-canonical shape (Fig. 2, *a*) from a large dark gray siltstone rock [15]. After the procedure of non-dimensionalization of the problem parameters, the analogy of which is presented in [20], the elastic characteristics of the material were $E_z = 6.21$, $E_r = 5.68$, $G_r = 2.29$, $G_z = 2.55$, $\nu_z = 0.22$, $\nu_r = 0.24$.

Let us investigate the distribution of stresses inside the area from the action of volumetric forces $\mathbf{X}^* = \{r^2, 0\}$ (Fig. 2, *b*), provided that the trace of these stresses on the boundary is zero $\mathbf{p}^* = 0$.

On the meridional section (see Fig. 2, *a*) negative coordinates along the r axis are designated conventionally, in reality $0 \le r \le 1$.

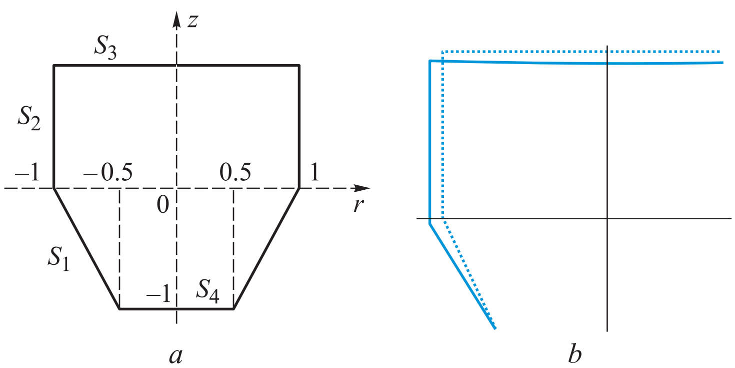


Fig. 2. The meridional section of the body (*a*) and the contour of the deformed body (*b*)

To solve the problem, a basis (12) of 70 elements was required; we will not present the Fourier coefficients. Checking the result and assessing the accuracy is carried out by comparing the specified boundary conditions with those reconstructed as a result of the solution. The volumetric forces in the areas S_1

and S_2 are shown in Fig. 3. In the Figures, the given (ı ı ı ı) and restored (———) volumetric forces are shown to scale. For example, the true R value in the left graph of Fig. 3, *a* is equal to the value on the graph multiplied by the coefficient κ.

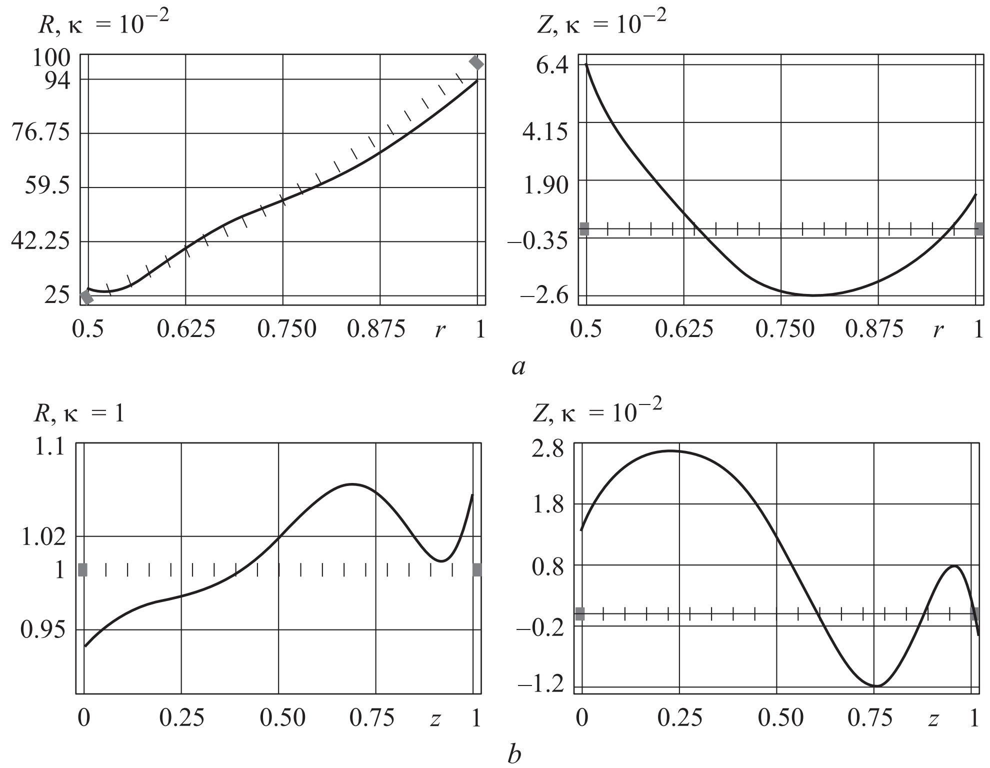


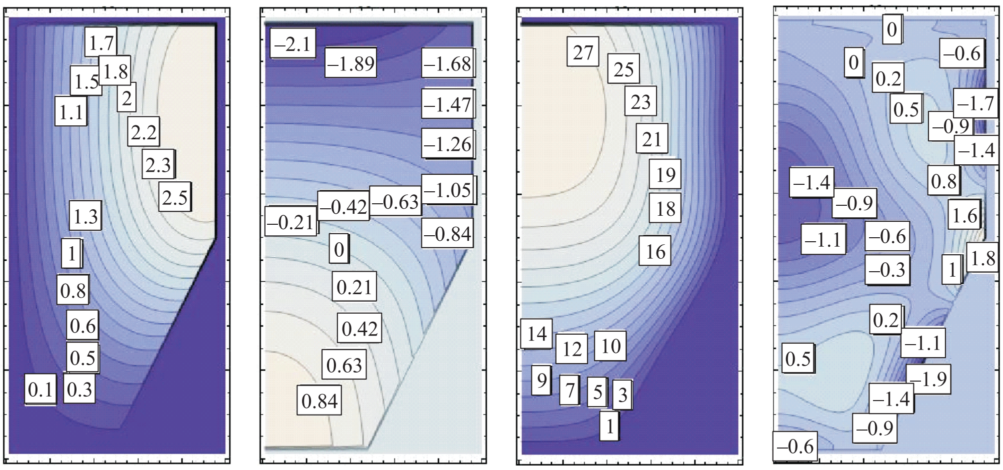
Fig. 3. Verification of volumetric forces at the border in sections S_1 (*a*) and S_2 (*b*)

The SSS components having a polynomial form are presented in the form of isolines (in an explicit form they are boundless) in Fig. 4 (for all isolines $\kappa = 10^{-2}$). Due to axial symmetry, the displayed region is $V_1 = \{(z, r) \mid \; 0 \le r \le 1, \; -1 \le z \le 1\}$.

The obtained elastic fields satisfy all equations of the elasticity theory for a transversely isotropic body [17].

Solutions of the second main, main mixed and mixed problem in the contact formulation are given in [21–23].

Results and discussion. The solution of problems in the linear theory of elasticity is constructed in the form of series, not relying entirely on the general or fundamental solution, as was done, for example, in [19]. Here the dependence of the vector of displacement of the plane auxiliary state on the coordinates $y^{\alpha} z^{\beta}$ is given and the displacement vector of the spatial axisymmetric state is determined on its basis. For such a vector, the deformation tensor is determined from the Cauchy relation, the stress tensor from Hooke's law, the forces on the surface of the body from the fundamental Cauchy relation, and the volume forces from the equilibrium equation. An exact particular solution of the problem is constructed, corresponding to the displacement function specified at each point of the body. Sorting out $\alpha + \beta \leq n$, $n = 1, 2, 3, \ldots$, a set of exact solutions is constructed and a basis is formed in the volumetric forces problem, which, together with the basis in the elastostatic problem, forms a set of particular solutions of the problem. Leaving only linearly independent ones among these solutions and carrying out their orthogonalization, we obtain a basis in which the corresponding vectors or tensors are expanded into series (6) with the same coefficients. Therefore, the outlined approach is broader than the approach based on general or fundamental decisions. In this case, the orthogonallization procedure is used, defined by relations (4) or (5), which allows immediately constructing the problem solution with given volumetric and surface forces, rather than a complete solution as the sum of a particular solution of the volumetric forces problem and the general solution of surface forces problem [19].



a *b* *c* *d*

Fig. 4. Isolines of SSS components:

a, b) displacement u, w; *c, d*) stresses σ_{rr}, σ_{zz}

However, the proposed method is not common for any class of considered domains (simply connected and multiply connected) and the type of given volume force functions. The convergence rate of the series depends on the boundary conditions and conditions inside the region, as well as on the geometry of the body.

Conclusion. The boundary state method has shown its effectiveness in solving classical boundary value problems of the elasticity theory for transversely isotropic bodies of revolution with volumetric forces. The solution has an analytical form, which makes it easy to analyze the obtained SSS characteristics.

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Ivanychev D.A. — Cand. Sc. (Eng.), Assist. Professor, Department of General Mechanics, Lipetsk State Technical University (Moskovskaya ul. 30, Lipetsk, 398055 Russian Federation).

Levina E.Yu. — Cand. Sc. (Phys.-Math.), Assist. Professor, Department of Physics, Bauman Moscow State Technical University (2-ya Baumanskaya ul. 5/1, Moscow, 105005 Russian Federation).

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