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RELIABILITY INTERVAL ESTIMATION FOR A SYSTEM MODEL WITH ELEMENT DUPLICATION IN DIFFERENT SUBSYSTEMS

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Abstract

The problem was considered of estimating reliability for a complex system model with element duplication of various subsystems and ensuring possibility of additional redundancy in a more flexible dynamic (or 'sliding') mode in each of the subsystems, which significantly increases reliability of the system in general. For the system considered, general model and analytical expressions were obtained in regard to the main reliability indicators, i.e., probability of the system failure-free operation (reliability function) for a given time and mean time of the system failure-free operation. On the basis of these analytical expressions, the lower confidence limit for the system reliability function was found in a situation, where the element reliability parameters were unknown, and only results of testing the system elements for reliability were provided. It was shown that the system resource function was convex in the reliability parameters vector of the system separate elements various types. Based on this, the lower confidence boundary construction for the system reliability function was reduced to the problem of finding the convex function extremum on a confidence set in the system element parameter space. In this case, labor consumption of the corresponding computational procedure increases linearly with an increase in the problem dimension. Numerical examples of calculating the lower confidence boundary for the system reliability function were provided

Keywords

Reliability, system, failure-free operation mean time, confidence boundary, redundancy

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Introduction. Estimation of the complex system reliability indicators based on test results of their separate components (elements, subsystems) is one of the urgent problems in the mathematical reliability theory. Currently, methods of interval estimation with a given validity level of the complex system reliability indicators based on results of their elements testing were elaborated mainly for classic sequential and sequential-parallel structural reliability schemes (see, for example, [1–10], etc.). Similar methods for solving the given problem for sequential-parallel models with independent (unlimited) recovery and loaded redundancy of elements are proposed in [11, 12]. Structures of the k type of n with redundancy in the loaded mode, which are some kind of natural generalization of the classic parallel structures, are considered in [13–15]. Further, a more general model of the system will be presented, where redundancy of elements is possible in its each subsystem both in the normal loaded mode, and with additional redundancy in a more flexible dynamic mode (when additional redundant elements are not rigidly connected to one or another main element of the original system). This makes it possible to significantly increase the system reliability (see examples). Such a model contains in a particular case sequential-parallel schemes with duplication (in the loaded mode) of the system elements. Exact analytical formulas were obtained expressing dependence of the system reliability function on the element reliability parameters. For the general model under consideration, solution to the problem often arising in engineering practice was proposed to construct a lower confidence boundary for the system reliability function based on results of testing its individual elements, which significantly expands the scope of existing methods for solving this problem.

Let us consider a system with sequential connection of the m different subsystems, where the i-th subsystem consists of the l sequentially connected elements with reliability function $P_i(t) = \exp(-\lambda_i t)$. Each element is duplicated by a single-type element with the same reliability function $P_i(t) = \exp(-\lambda_i t)$, where $\lambda_i > 0$ is the failure rate parameter for the i-th type elements (i-th subsystem), $i = 1, ..., m$. Thus, the i-th subsystem appears to be a sequential connection of the l_i subsystems with duplication of the main element with reliability function $1 - [1 - P_i(t)]^2$, $i = 1, ..., m$ (redundancy mode is assumed to be loaded). In addition, each i-th subsystem is provided additionally with r_i reserve elements of the i-th type staying in the dynamic redundancy mode, when each of these r_i reserve elements is not rigidly attached to one or another fixed main element of the i-th subsystem. And in case of any element failure, it is replaced in the i-th subsystem by one of the additional r_i backup elements, $i = 1, ..., m$.

Under the assumption that failures of the system various elements occur independently of each other, probability of failure-free operation (reliability function) of the system over the time interval $(0, t)$ is having the following form:

$$P_{sys}(\lambda, t) = \prod_{i=1}^{m} H(l_i, r_i, \lambda_i t), \quad (1)$$

where $H(l_i, r_i, \lambda_i t)$ is the i-th subsystem reliability function, $i = 1, ..., m$; $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is the reliability parameters vector of system elements. In many cases, exact values of the $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ element parameters are unknown, and only the results of testing the system elements for reliability are provided, and it is required to construct a lower confidence boundary for the system reliability function $P_{sys}(\lambda, t)$.

System reliability function calculation. Let us consider calculation of the system reliability function (1) under assumption that the $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ reliability parameters are known. Let us denote by $H(l, r, \lambda t)$ the reliability function for a single separate subsystem of the above type consisting of l subsystems with duplication and of r additional redundant elements (in the sliding redundancy mode). Then the η failure-free operation time of such a subsystem is a convolution of two independent random variables:

$$\eta = \xi_1 + \xi_2, \quad (2)$$

where ξ_1 is the failure-free operation time of a system of the $(2l+1)$ type from $(2l+r)$, i.e., a system that consists of $(2l+r)$ elements with the $P(t) = \exp(-\lambda t)$ reliability function is faultless, if at least two $(2l+1)$ of them are serviceable. For the ξ_1 random variable, corresponding reliability function has the following form [1]:

$$P_1(t) = P\{(\xi_1) > t\} = \sum_{j=k}^{n} C_n^j p^j (1-p)^{n-j}, \quad (3)$$

where $n = 2l + r$; $k = 2l + 1$; $p = \exp(-\lambda t)$. Hence, for this model

$$P_1(t) = \sum_{j=k}^{n} C_n^j e^{-j\lambda t} (1 - \exp(-\lambda t))^{n-j}. \quad (4)$$

The ξ_2 random variable is the failure-free time for initial structure of the l sequentially connected parallel subsystems of two elements (excluding the additional r redundant elements) with the reliability function $P_2(t) = P\{\xi > t\} =$ $= \{1 - [1 - \exp(-\lambda t)^2]\}^l$. In accordance with (2), the $H(l, r, \lambda t)$ system reliability function is having the following form:

$$H(l, r, \lambda t) = \int_0^t f_1(u) P_2(t-u)\, du + P_1(t). \quad (5)$$

Here $f_1(u) = -P_1'(u)$ is distribution density of the ξ_1 random variable. After simple transformations from (3) and (4), it follows:

$$f_1(t) = \lambda k C_n^k \exp(-k\lambda t)[1-\exp(-\lambda t)]^{r-1}, \tag{6}$$

where $n = 2l + r$; $k = 2l + 1$; $l \geq 1$; $r \geq 1$. Based on the expression for the system reliability function (5) and formula (6), the following is found:

$$I_t = \int_0^t f_1(u) P_2(t-u)\, du =$$

$$= \lambda k C_n^k \int_0^t e^{-k\lambda u}(1-\exp(-\lambda u))^{r-1}(2\exp(-\lambda(t-u)) - \exp(-2\lambda(t-u)))^l\, du.$$

After simple transformations:

$$I_t = \lambda k C_k^n 2^l \exp(-l\lambda t) \int_0^t (\exp(-\lambda u))^{l+1}(1-\exp(-\lambda u))^{r-1} g_l(t,u)\, du, \tag{7}$$

where

$$g_l(t,u) = \left[1 - \frac{1}{2}\exp(-\lambda(t-u))\right]^l. \tag{8}$$

Considering that

$$(1-\exp(-\lambda u))^{r-1} = \sum_{j=0}^{r-1} C_{r-1}^j (-1)^j \exp(-j\lambda u),$$

it follows from (7) and (8):

$$I_t = \lambda k C_n^k \exp(-l\lambda t) \sum_{j=0}^{r-1} (-1)^j C_{r-1}^j \int_0^t h_l(t,u)\, du. \tag{9}$$

Here $h_l(t,u) = g_l(t,u)\exp[-(l+j+1)\lambda u]$.

In accordance with (8), the following expression satisfies function $g_l(t,u)$:

$$g_l(t,u) = \sum_{i=0}^{l} (-1)^i \left(\frac{1}{2}\right)^i C_l^i \exp(-i\lambda(t-u)). \tag{10}$$

Further, it is found from (5)–(10) that the $H(l,r,\lambda t)$ system reliability function for the model under consideration is having the following form:

$$H(l,r,\lambda t) = P_1(t) + k C_n^k 2^l \sum_{i=0}^{l} \frac{(-1)^i C_l^i}{2^i} \sum_{j=0}^{r-1} \frac{(-1)^j C_{r-1}^j B_{ij}(\lambda t)}{l-i+j+1}, \tag{11}$$

where $B_{ij}(\lambda t)$ are functions,

$$B_{ij}(\lambda t) = \exp(-(l+i)\lambda t) - \exp(-(2l+j+1)\lambda t), \qquad (12)$$

$l \geq 1;\ r \geq 1;\ k = 2l+1;\ n = 2l+r.$ For the given model and based on (11), appropriate expression could be found for the system failure-free operation mean time:

$$\mu_{sys} = \mu_{sys}(l, r, \lambda) = \int_0^\infty H(l, r, \lambda t)dt.$$

For the k type structure of n with the $P_1(t)$ reliability function, the average failure-free time is determined by the following expression [1, 3]:

$$\mu_1 = \int_0^\infty P_1(t)\,dt = \frac{1}{\lambda}\sum_{j=k}^{n}\left(\frac{1}{j}\right). \qquad (13)$$

According to (12):

$$\int_0^\infty B_{ij}(\lambda t)\,dt = \frac{l+j-i+1}{\lambda(l+i)(2l+i+1)}. \qquad (14)$$

It is found for the model under consideration from (11)–(14) that the average failure-free operation time is determined as

$$\mu_{sys}(l, r, \lambda) = \sum_{j=k}^{k}\frac{1}{i\lambda} + kC_n^k 2^l \sum_{i=0}^{l}\frac{(-1)^i C_l^i}{2^i(l+i)}\sum_{j=0}^{r-1}\frac{(-1)^j C_{r-1}^j}{(2l+j+1)\lambda},$$

where $k = 2l+1;\ n = 2l+r.$

Lower confidence boundary construction for the system reliability function. Let us consider the problem of the system reliability estimation in case, when the $\lambda = (\lambda_1, ..., \lambda_m)$ elements reliability parameters are unknown and are determined by results of their testing. Further, we assume that testing elements of the *i*-th type system were carried out according to the standard plans of the $[N_i B T_i]$ type in designations given in [1], i.e., N_i elements of the *i*-th type were exposed to testing (with restoration of the failed elements) over the T_i time; as a result, d_i failures were registered $i = 1, ..., m.$ It should be noted that in case of highly reliable elements, i.e., insignificant failures, test plans with and without restoration of the failed elements are approximately equivalent [1]. It is required, based on the test results vector $d = (d_1, ..., d_m),$ to construct the $\underline{P}_{sys} = \underline{P}_{sys}(d, t)$ lower γ-confidence boundary for the system reliability function (1).

Let us denote by $D = d_1 + ... + d_m$ the total number of the element failures. Since the d_i random variable has Poisson distribution with the $\Lambda_i = N_i T_i \lambda_i,$ $i = 1, ..., m,$ parameter (see [1–4]), the D random variable also has Poisson distribution with parameter $\Lambda = \sum_{i=1}^{m} N_i T_i \lambda_i$. Let us denote by $\Delta_\gamma(d)$ the standard upper γ-confidence boundary for the Poisson distribution law parameter [1, 16]. Then, the following inequity is performed in accordance with definition of this boundary:

$$P\left\{ \sum_{i=1}^{m} N_i T_i \lambda_i \le \Delta_\gamma(D) \right\} \ge \gamma, \tag{15}$$

here γ is the confidence coefficient. For each $d = (d_1, ..., d_m)$ vector of test results, let us introduce the G_d subset in the parameter space $\lambda = (\lambda_1, ..., \lambda_m)$, which is provided by the following inequalities:

$$\sum_{i=1}^{m} N_i T_i \lambda_i \le \Delta_\gamma(D), \quad \lambda_i \ge 0, \quad i = 1, ..., m. \tag{16}$$

It follows from (15) that the $d = (d_1, ..., d_m)$ sets determined in this way generate a system of γ-confidence sets for the vector of parameters $\lambda = (\lambda_1, ..., \lambda_m)$. In accordance with the general method of confidence sets (see [1, 5, 16], etc.), the $\underline{P}_{sys} = \underline{P}_{sys}(d, t)$ lower γ-confidence boundary for the system reliability function could be found as:

$$\underline{P}_{sys} = \underline{P}_{sys}(d, t) = \min P_{sys}(\lambda, t). \tag{17}$$

The minimum is taken here over all $\lambda = (\lambda_1, ..., \lambda_m) \in G_d$, i.e., over the region given by inequalities (16). The system reliability function could be represented as:

$$P_{sys}(\lambda, t) = \prod_{i=1}^{m} H(l_i, r_i, \lambda_i t) = \exp[-f(\lambda, t)], \tag{18}$$

where

$$f(\lambda, t) = \sum_{i=1}^{m} f_i(l_i, r_i, \lambda_i t); \tag{19}$$

$$f_i(l_i, r_i, \lambda_i t) = -\ln H(l_i, r_i, \lambda_i t). \tag{20}$$

In accordance with expressions (17)–(20), the lower confidence limit for the system reliability function in (17) has the following form:

$$\underline{P}_{sys}(d, t) = \exp\left(-\overline{f}(d, t)\right). \tag{21}$$

Here $\overline{f}(d, t)$ is the upper confidence bound for function $f(\lambda, t)$,

$$\overline{f}(d,t) = \max f(\lambda, t), \tag{22}$$

where the maximum is taken by $\lambda \in G_d$. Let us show that the $f(\lambda, t)$ function is convex in the vector of parameters $\lambda = (\lambda_1, ..., \lambda_m)$, which significantly simplifies the problem of finding the maximum in (22). For this purpose and in accordance with expressions (19), (20), it is sufficient to demonstrate that each i-th subsystem with the $H(l_i, r_i, \lambda_i t)$ reliability function is a subsystem with the failure increasing rate function (IRF-subsystem). This is equivalent to each $f_i(l_i, r_i, \lambda_i t)$ function convexity in $\lambda_i t$, and, therefore, in the λ_i parameter for any fixed t.

Convolution of the ξ_1, ξ_2, independent random variables, each of which is having the IRF distribution (with the increasing function of failure rate) also is provided with the IRF distribution [3]. Thus, taking into account relation (2), it is sufficient further to show that each of the ξ_1, ξ_2, random variables has the IRF distribution. For random variable ξ_2, it is not difficult to demonstrate by direct differentiation that the function

$$\Lambda_2(t) = -\ln P_2(t) = -\ln(1 - [1 - \exp(-\lambda t)]^2)^l$$

has a monotonically increasing derivative. It follows from this that the ξ_2 random variable has IRF distribution. It follows for the $P_1(t)$ reliability function of the ξ_1 random variable from expression (3) that:

$$\frac{d}{dt}[-\ln P_1(t)] = \frac{d}{dp}\left[-\ln \sum_{j=k}^{n} C_n^j p^j q^{n-j}\right]\frac{dp}{dt},$$

where $p = \exp(-\lambda t)$; $q = 1 - \exp(-\lambda t)$, from where

$$\frac{d}{dt}[-\ln P_1(t)] = \lambda C_n^k p^k q^{n-k}\left[\sum_{j=k}^{n} C_n^j p^j q^{n-j}\right]^{-1} =$$

$$= \lambda C_n^k \left[\sum_{j=k}^{n} C_n^j (\exp(\lambda t) - 1)^{k-j}\right]^{-1}.$$

This function is monotonically increasing in t and, therefore, the ξ_1 random variable also has the IRF distribution. Thus, the $f_i(l_i, r_i, \lambda_i t)$ function for each i-th subsystem determined in (20) is convex with respect to parameter $\lambda_i > 0$. So, the $f(\lambda, t)$ function is convex with respect to the vector of all parameters $\lambda = (\lambda_1, ..., \lambda_m)$. In accordance with known results of the convex programming theory [17], the convex function maximum in (22) is reached in one of the G_d

region 'corner' points of form $(0,...,0,\tilde{\lambda}_i,0,...,0)$, where $\tilde{\lambda}_i = \Delta_\gamma(D)/(N_i T_i)$, $i=1,...,m.$ It follows that the upper confidence boundary (22) for the $f(\lambda t)$ function is determined by the following expression:

$$\overline{f}(d,t) = \max_i f_i(l_i, r_i, \tilde{\lambda}_i t) = \max_i f_i\,[l_i, r_i, \Delta_\gamma(D)t/(N_i T_i)],$$

where the maximum is taken over the entire $i=1,..., m.$ Whence and taking into account (21), the corresponding expression follows for the lower confidence boundary of the system reliability function:

$$\underline{P}_{sys}(d,t) = \min_i H_i(l_i, r_i, \tilde{\lambda}_i t) = \min_i H_i[l_i, r_i, \Delta_\gamma(D)t/(N_i T_i)].$$

Let us consider further several numerical examples illustrating construction of the lower confidence boundary for the system reliability function.

Example 1. The system consists of $m=5$ various subsystems. Parameters of the l_i, r_i subsystems and test results of various type elements N_i, T_i, d_i, $i=1, ..., m,$ are provided below:

i	1	2	3	4	5
l_i	2	5	2	4	3
r_i	2	5	2	4	3
N_i	10	12	10	12	15
T_i	50	15	20	25	17
d_i	1	0	0	1	0

Let us consider the case of ordinary loaded redundancy, where each r_i back-up element in the i-th subsystem reserves one or another fixed element in this subsystem $i=1,...,m.$ In this case, the lower confidence boundary ($\gamma = 0.95$ confidence factor) for the system reliability function (at $t = 20$) is $\underline{P}_{sys}(d, t) = 0.891.$

Example 2. Under conditions provided in Example 1, let us consider the case, where in accordance with the above model, each i-th subsystem contains r_i reserve elements staying in the dynamic redundancy mode, $i=1,...,m.$ Then, the lower confidence boundary for the system reliability function constructed with the same data and using the above formulas is $\underline{P}_{sys}(d,t) = 0.971.$ This value compared to the value obtained in Example 1 shows a significant gain from using the proposed approach (with the same confidence factor).

Conclusion. Precise analytical formulas were found for the general system model with duplication of elements and possible additional redundancy in a more flexible element dynamic mode in each of the subsystems. These formulas indicate dependence of the system reliability function and the average time of failure-free operation on the element reliability parameters. On the basis of these analytical expressions, solution to the problem of constructing the lower confidence boundary for the system reliability function was also obtained, based on results of testing its elements. Further, confidence estimations could be built on this basis for such major indicators as probability of the system failure-free operation for a given time, γ-percentage resource and average failure-free system operation time. Numerical examples were considered illustrating the benefits of the proposed approach in calculating the lower confidence boundary for the system reliability function. From the point of view of applications, extension of the results presented in regard to more general models of complex systems, as well as to more general classes (including nonparametric) of the system element failure-free operation time distribution, remains of considerable interest.

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