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TOPOLOGICAL GRADING OF SEMIGROUP C^*-ALGEBRAS

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Abstract

The paper deals with the abelian cancellative semigroups and the reduced semigroup C^*-algebras. It is supposed that there exist epimorphisms from the semigroups onto the group of integers modulo n. For these semigroups we study the structure of the reduced semigroup C^*-algebras which are also called the Toeplitz algebras. Such a C^*-algebra can be defined for any non-abelian left cancellative semigroup. It is a very natural object in the category of C^*-algebras because this algebra is generated by the left regular representation of a semigroup. In the paper, by a given epimorphism σ we construct the grading of a semigroup C^*-algebra. To this aim the notion of the σ-index of a monomial is introduced. This notion is the main tool in the construction of the grading. We make use of the σ-index to define the linear independent closed subspaces in the semigroup C^*-algebra. These subspaces constitute the C^*-algebraic bundle, or the Fell bundle, over the group of integers modulo n. Moreover, it is shown that this grading of the reduced semigroup C^*-algebra is topological. As a corollary, we obtain the existence of the contractive linear operators that are non-commutative analogs of the Fourier coefficients. Using these operators, we prove the result on the geometry of the underlying Banach space of the semigroup C^*-algebra

Keywords

Cancellative semigroup, grading, monomial, reduced semigroup C^*-algebra, topologically graded C^*-algebra, σ-index

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Introduction. In the paper we study the structure of the reduced semigroup C^*-algebras. These algebras are generated by the left regular representations of semigroups with the cancellation property. The study of such C^*-algebras was started in [1–3]. The theory of semigroup C^*-algebras was further developed in the papers by a number of authors (see literature, for example, in [4–6]).

As is well-known, a grading for an object of a category sheds light on the structure of this object. In the category of C^*-algebras, one considers the C^*-gradings that are also called the C^*-bundles, or the Fell bundles. These bundles were introduced in [7] in order to generalize the concepts of harmonic analysis for the non-commutative case. The notion of the topologically graded C^*-algebra was introduced in [8]. An important property of such an algebra is the existence of special operators, namely, the conditional expectation and the Fourier coefficients. We refer the reader to [9] for a detailed presentation of the theory of graded C^*-algebras.

This paper deals with the problem on a construction of gradings for the reduced semigroup C^*-algebras. The work continues studies on graded semigroup C^*-algebras started in [10–13]. The main result of the paper is the following statement (Theorem 2). If there exists a semigroup epimorphism $\sigma: L \to Z_n$ from an abelian cancellative semigroup L onto the group Z_n of integers modulo n, then the semigroup C^*-algebra $C_r^*(L)$ is a topologically Z_n-graded C^*-algebra. To construct the indicated grading, the second author of the paper has proposed introducing the notion of the σ-index of a monomial. As a consequence of the above-mentioned result, one obtains the statement announced in [13, Theorem 5] on the topological grading of the C^*-algebra $C_r^*(L)$ for a semigroup L, which is a normal extension of a certain semigroup.

Preliminaries. Let L be a discrete abelian additive semigroup with the cancellation property. Its neutral element will be denoted by the symbol 0.

The object of the study in the present paper is the reduced semigroup C^*-algebra $C_r^*(L)$. Now we recall the definition of this algebra.

Let us consider the Hilbert space of all square summable complex-valued functions defined on the semigroup L:

$$l^2(L) := \left\{ f: L \to C \mid \sum_{y\in L} |f(y)|^2 < +\infty \right\}.$$

The canonical orthonormal basis of the Hilbert space $l^2(L)$ is denoted by $\{ e_y \mid y \in L \}$, where

$$e_y(x) := \begin{cases} 1 & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases}$$

The reduced semigroup C^*-algebra $C_r^*(L)$ is the C^*-subalgebra in the algebra of all bounded operators on the Hilbert space $l^2(L)$ generated by the set of isometries $\{T_y \mid y \in L\}$. Here the operator T_y is defined as follows: $T_y(e_x) = e_{x+y}$, $x, y \in L$.

As usual, let $Z_n = Z/nZ$ be the additive group of integers modulo n. The elements of the group Z_n are denoted by the symbols $[0]_n, [1]_n, ..., [n-1]_n$.

In what follows, we will assume that there is a surjective semigroup homomorphism

$$\sigma : L \to Z_n. \tag{1}$$

Then the semigroup L can be represented as the disjoint union of its subsets

$$L = L_0 \bigcup L_1 \bigcup ... \bigcup L_{n-1}, \tag{2}$$

where each L_k is the preimage of the element $[k]_n \in Z_n$ under σ, that is, $\sigma^{-1}([k]_n) = L_k$, $0 \le k \le n-1$.

Further, for the convenience of the reader, we recall the definition of a G-graded C^*-algebra from [9, § 16.2].

Let $\mathbf{A}$ be an arbitrary C^*-algebra and G be a group. The algebra $\mathbf{A}$ is said to be a G-graded C^*-algebra, if there exists a family of linearly independent closed subspaces $\{\mathbf{A}_g \subset \mathbf{A}\}_{g \in G}$ such that for any $g, h \in G$ the following properties are satisfied:

1) $\mathbf{A}_g \mathbf{A}_h \subset \mathbf{A}_{gh}$;

2) $\mathbf{A}_g^* = \mathbf{A}_{g^{-1}}$;

3) $\mathbf{A} = \overline{\bigoplus_{g \in G} \mathbf{A}_g}$.

Moreover, the family of Banach spaces $\{\mathbf{A}_g\}_{g \in G}$ is called a *C^*-algebraic bundle*, or a *Fell bundle*, over the group G.

The notion of the grading in a stronger sense was considered in [9, § 19.2]. Namely, a G-graded C^*-algebra $\mathbf{A}$ is said to be *topologically graded*, if there exists a bounded linear operator F: $\mathbf{A} \to \mathbf{A}$, which has the following properties. It is the identity operator on the subspace $\mathbf{A}_e$, where e is the unit of the group, and it vanishes on each subspace $\mathbf{A}_g$, where $g \ne e$. The original definition of the topologically graded C^*-algebra [9, Definition 19.2] contains a stronger requirement. It assumes that the operator F is a conditional expectation on the subspace $\mathbf{A}_e$. However, as was noted in [14], this requirement is satisfied automatically. This fact follows from Theorem 19.1 given in [9].

It is worth noting that an example of a G-graded C^*-algebra, which is not topologically graded, is contained in [9, § 19.3].

As was noted in Introduction, an important property of a topologically graded C^*-algebra is the existence of the Fourier coefficients (see [9, § 19.6]). This means that for each $g \in G$ there exists a contractive linear operator $F_g : \mathbf{A} \to \mathbf{A}_g$ such that the equality $F_g(A) = A_g$ holds for every finite sum $A = \sum_{h \in G} A_h$, where $A_h \in \mathbf{A}_h$.

Grading of the C^*-algebra $C_r^*(L)$. To construct the grading of the semigroup C^*-algebra $C_r^*(L)$ we introduce the concepts of the monomial, the σ-index of monomial and the operator in this C^*-algebra corresponding to a monomial, where σ is surjective semigroup homomorphism (1).

Firstly, we consider the free semigroup with the set of generators (the alphabet) $\left\{T_y^0, T_y^1 \mid y \in L\right\}$. The elements of this semigroup are the words of the following form:

$$V = T_{y_k}^{i_k} T_{y_{k-1}}^{i_{k-1}} \ldots T_{y_1}^{i_1}, \qquad (3)$$

where $y_1, \ldots, y_k \in L$, $i_1, \ldots, i_k \in \{0, 1\}$.

We call these words *the monomials*. The number k in (3) is called *the monomial length.* The free semigroup is said to be *the monomial semigroup*, and it is denoted by Mon.

Let us define the involution on Mon as follows. For a monomial of form (3) we set

$$V^* = T_{y_1}^{1-i_1} T_{y_2}^{1-i_2} \ldots T_{y_k}^{1-i_k}.$$

Thus, the monomial semigroup Mon becomes an involutive semigroup.

Secondly, we define the mapping $\operatorname{ind} : \mathrm{Mon} \to \mathbf{Z}_n$. To do this, for a monomial of form (3), we put by definition:

$$\operatorname{ind} V = (-1)^{i_k} \sigma(y_k) + \ldots + (-1)^{i_1} \sigma(y_1).$$

It is easy to see that for any $V, W \in \mathrm{Mon}$ the following equalities hold:

$$\operatorname{ind}(V \cdot W) = \operatorname{ind} V + \operatorname{ind} W;$$

$$\operatorname{ind}(V^*) = -\operatorname{ind} V.$$

Consequently, the mapping ind is an involutive surjective homomorphism of semigroups.

Thirdly, to each monomial V we assign the operator $\hat{V}$ on the Hilbert space $l^2(L)$ by the rules:

$$\hat{T}_y^0 = T_y, \quad \hat{T}_y^1 = T_y^*$$

and if V has form (3), then

$$\hat{V} = \hat{T}_{y_k}^{i_k} \hat{T}_{y_{k-1}}^{i_{k-1}} \dots \hat{T}_{y_1}^{i_1}. \tag{4}$$

Lemma 1. *Let* $V \in \mathrm{Mon}$ *and* $\hat{V}e_y \neq 0$ *for a basis vector* $e_y \in l^2(L)$. *Then the following equalities hold:* $\hat{V}e_y = e_z$ *and* $\sigma(z) = \sigma(y) + \mathrm{ind}\, V$.

◄ Let $V \in \mathrm{Mon}$. We consider representation (3) of this monomial. Further, we prove the assertion of the lemma by induction on the monomial length.

Let $k = 1$. Then there are two possibilities: $V = T_{y_1}^0$ or $V = T_{y_1}^1$. Let $\hat{V}e_y \neq 0$, where $y \in L$. In the first case, we obtain the equality $T_{y_1}e_y = e_{y+y_1}$. Hence, we have $z = y + y_1$ and

$$\sigma(z) = \sigma(y) + \sigma(y_1) = \sigma(y) + \mathrm{ind}\, T_{y_1}^0.$$

In the second case, we get $T_{y_1}^* e_y = e_z$, where $z + y_1 = y$ and $\sigma(z) + \sigma(y_1) = \sigma(y)$. Consequently, we have the following equalities:

$$\sigma(z) = \sigma(y) - \sigma(y_1) = \sigma(y) + \mathrm{ind}\, T_{y_1}^1.$$

Now, we assume that the assertion of the lemma is true for every monomial V' of the length $k-1$, i.e., if $\hat{V}'e_y \neq 0$ then $\hat{V}'e_y = e_{z'}$ and the equality $\sigma(z') = \sigma(y) + \mathrm{ind}\, V'$ holds.

Next, let us consider an arbitrary monomial V of the length k. Obviously, it can be written in the form $V = T_{y_k}^{i_k} V'$. Then we get $\hat{V}e_y = \hat{T}_{y_k}^{i_k} e_{z'} = e_z$. In this case, as in the case when $k = 1$, one obtains the equality $\sigma(z) = \sigma(z') + \mathrm{ind}\, T_{y_k}^{i_k}$. Thus, using the induction hypothesis, we have the required equality:

$$\sigma(z) = \sigma(y) + \mathrm{ind}\, V' + \mathrm{ind}\, T_{y_k}^{i_k} = \sigma(y) + \mathrm{ind}\, V. \quad ►$$

It follows from Lemma 1 that if $\hat{V}_1 = \hat{V}_2$ then $\mathrm{ind}\, V_1 = \mathrm{ind}\, V_2$. Further, for a monomial $V \in \mathrm{Mon}$, the value $\mathrm{ind}\, V$ of the mapping ind at V will be called both the σ-*index* of the monomial V and the σ-index of the operator $\hat{V}$.

We note that all finite linear combinations of operators of form (4) constitute a dense involutive subalgebra in the C^*-algebra $C_r^*(L)$. Let us denote this subalgebra by $P(L)$.

It is easy to see that all monomials with the σ-indexes $[0]_n$ constitute an involutive subsemigroup in the monomial semigroup Mon. In the C^*-algebra $C_r^*(L)$ we consider the C^*-subalgebra $\mathbf{A}_0$ generated by all operators of form (4) with the σ-index $[0]_n$. Moreover, we define the Banach subspace $\mathbf{A}_k$ in $C_r^*(L)$ for every integer k, where $1 \le k \le n-1$. This space is the closure for the linear span of all operators of form (4) with the σ-index $[k]_n$, $1 \le k \le n-1$.

Further, using decomposition (2), we represent the Hilbert space $l^2(L)$ as the orthogonal sum of its subspaces:

$$l^2(L) = \bigoplus_{k=0}^{n-1} H_k, \tag{5}$$

where for each H_k the family of functions $\{e_y \mid y \in L_k\}$ is an orthonormal basis.

The following lemma shows behaviors of the spaces H_k under actions of the elements of the spaces $\mathbf{A}_k$.

Lemma 2. *For every* $k, l \in \mathbf{N}$ *such that* $0 \le k, l \le n-1$ *and for every operator* $A \in \mathbf{A}_k$ *the following property holds:* $A(H_l) \subset H_r$, *where* $[r]_n = [k+l]_n$. *In particular, for each integer k satisfying the inequality* $0 \le k \le n-1$ *the subspace* H_k *is invariant under the action of every element of the* C^**-algebra* $\mathbf{A}_0$.

◄ Since the finite linear combinations of operators having form (4) with the σ-index $[k]_n$ constitute a dense subspace in the Banach space $\mathbf{A}_k$ it is sufficient to prove the statement of the lemma for such operators.

Let V be a monomial with the σ-index $[k]_n$. Take a vector $e_y \in H_l$. Thus we have the equality $\sigma(y) = [l]_n$. Then, by Lemma 1, the following property is fulfilled. If $\hat{V} e_y \ne 0$ then $\hat{V} e_y = e_z$ for $z \in L$ and the following equality is true: $\sigma(z) = \sigma(y) + \operatorname{ind} V$. Thus we obtain the equality $\sigma(z) = [l]_n + [k]_n$. Therefore we have $e_z \in H_r$, where $[r]_n = [k+l]_n$. ►

Further we use Lemma 2 for proving the following lemma, which states that the family of subspaces $\{\mathrm{A}_k \mid 0 \le k \le n-1\}$ is the C^*-algebraic bundle over the group $\mathbf{Z}_n$.

Lemma 3. *For the family of subspaces* $\{\mathrm{A}_k \mid 0 \le k \le n-1\}$ *the following statements hold:*

1) $\mathbf{A}_k \mathbf{A}_l \subset \mathbf{A}_m$, *where* $[m]_n = [k+l]_n$;

2) $\mathbf{A}_k^* = \mathbf{A}_m$, *where* $[m]_n = [n-k]_n$;

3) the family $\{\mathrm{A}_k \mid 0 \le k \le n-1\}$ *consists of the linearly independent closed subspaces of the* C^**-algebra* $C_r^*(L)$;

4) $C_r^*(L) = \overline{\bigoplus_{k=0}^{n-1} \mathbf{A}_k}$.

◄ It suffices to prove statements 1) and 2) for an operator of form (4) with the σ-index $[k]_n$ because the set of all finite linear combinations of such operators is dense in the Banach space $\mathbf{A}_k$.

1. Let V and W be monomials with the σ-indexes $[k]_n$ and $[l]_n$, respectively. Then the following equalities hold: $\operatorname{ind}(VW) = \operatorname{ind} V + \operatorname{ind} W = [k]_n + [l]_n = [k+l]_n$. Therefore we have $\hat{V}\hat{W} \in \mathbf{A}_m$, where $[m]_n = [k+l]_n$.

2. Let $V \in \mathrm{Mon}$ and $\hat{V}e_y \neq 0$ for a basis vector $e_y \in l^2(L)$. Then, by Lemma 1, we have $\hat{V}e_y = e_z$ for some $z \in L$, and the equality $\operatorname{ind} V = \sigma(z) - \sigma(y)$ holds. It is evident that $\hat{V}^* e_z = e_y$ and, by the same lemma, we obtain the relation $\operatorname{ind} V^* = \sigma(y) - \sigma(z) = -\operatorname{ind} V$. That is, if V is a monomial with the σ-index $[k]_n$ then we get $\operatorname{ind} V^* = -[k]_n = [n-k]_n$.

3. Let $A = \sum_{k=0}^{n-1} A_k = 0$, where $A_k \in \mathbf{A}_k$. We will show that this condition implies the equality $A_k = 0$ for every k. To this end, we fix an integer k, $0 \le k \le n-1$. Obviously, we have

$$A_k = -\sum_{j=0, j\neq k}^{n-1} A_j. \tag{6}$$

By Lemma 2, for every integer l satisfying the inequality $0 \le l \le n-1$ we get the property $A_k(H_l) \subset H_r$, where $[r]_n = [k+l]_n$. This inclusion implies the following condition:

$$-\sum_{j=0, j\neq k}^{n-1} A_j(H_l) \subset H_r. \tag{7}$$

On the other hand, again by Lemma 2, we have the inclusion

$$-\sum_{j=0, j\neq k}^{n-1} A_j(H_l) \subset \bigoplus_{j=0, j\neq k}^{n-1} H_{r_j}, \tag{8}$$

where $[r_j]_n = [j+l]_n$. We claim that the following equality holds:

$$H_r \bigcap \bigoplus_{j=0, j\neq k}^{n-1} H_{r_j} = \{0\}. \tag{9}$$

Indeed, for the sake of contradiction, we assume that equality (9) is not true. Using the orthogonality of the subspaces H_k, $0 \le k \le n-1$, we have the equality $H_r = H_{r_j}$ for some j. That is, we have the equalities $r = r_j$ and $[k+l]_n = [j+l]_n$.

Hence, we get $[k]_n = [j]_n$. Since the inequalities $0 \le k,\ j \le n-1$ hold, we have the equality $k = j$. Thus we have the contradiction. Therefore, equality (9) is valid, as claimed. Now, by equalities (6)–(9), we obtain the condition $A_k = 0$.

4. The set of all finite linear combinations of operators of form (4) is dense in the C^*-algebra $C_r^*(L)$. Moreover, this set is also dense in $\bigoplus_{k=0}^{n-1} \mathbf{A}_k$, because each such finite linear combination can be represented in the form $\sum_{k=0}^{n-1} A_k$, where A_k is a finite linear combination of operators of form (4) with the σ-index $[k]_n$. It follows from the relation $\bigoplus_{k=0}^{n-1} \mathrm{A}_k \subset C_r^*(L)$ that the space $\bigoplus_{k=0}^{n-1} \mathrm{A}_k$ is dense in $C_r^*(L)$. ►

Lemma 3 allows us to formulate the following result on the $\mathbf{Z}_n$-grading of the semigroup C^*-algebra $C_r^*(L)$.

Theorem 1. *Let* $\sigma : L \to \mathrm{Z}_n$ *be a surjective homomorphism of semigroups. Let* $\mathbf{A}_k$ *be the closed subspace in the* C^**-algebra* $C_r^*(L)$ *generated by all operators of form (4) with the* σ*-index* $[k]_n$, *where* $0 \le k \le n-1$. *Then the family of subspaces* $\{\mathrm{A}_k \mid 0 \le k \le n-1\}$ *constitutes the Fell bundle for the semigroup* C^**-algebra* $C_r^*(L)$ *over the group* $\mathbf{Z}_n$.

Topological grading of the C^*-algebra $C_r^*(L)$. Here we first prove that the grading of the semigroup C^*-algebra $C_r^*(L)$ constructed in the previous section is topological. At the end of the paper we prove the statement on the geometry of the underlying Banach space of the reduced semigroup C^*-algebra. Namely, it is shown that the direct sum for the family of the closed subspaces $\{\mathrm{A}_k \mid 0 \le k \le n-1\}$ forming the Fell bundle of the semigroup C^*-algebra $C_r^*(L)$ over the group $\mathbf{Z}_n$ coincides with $C_r^*(L)$, i.e., the following equality holds:

$$C_r^*(L) = \bigoplus_{k=0}^{n-1} \mathbf{A}_k.$$

In the proof of the following statement, we construct a linear operator that allows us to talk about the topological $\mathbf{Z}_n$-grading of the semigroup C^*-algebra $C_r^*(L)$.

Lemma 4. *There exists a contractive linear operator* $F : C_r^*(L) \to C_r^*(L)$, *which is the identity mapping on* $\mathbf{A}_0$ *and vanishes on each subspace* $\mathbf{A}_k$, *where* $1 \le k \le n-1$.

◄ We recall that the involutive subalgebra $P(L)$ consisting of all finite linear combinations of operators of form (4) is dense in the C^*-algebra $C_r^*(L)$. Therefore, according to the principle of the extension by continuity (see, for example, [15, Ch. 2, § 1, Theorem 2]) it suffices to construct a linear bounded operator $F: P(L) \to C_r^*(L)$, which satisfies the following two conditions. First, F is the identity operator on the linear combinations of operators of form (4) with the σ-index $[0]_n$. Second, F vanishes on the linear combinations of operators of form (4) with the σ-index $[k]_n$ for each integer k such that $1 \le k \le n-1$.

Since each element $A \in P(L)$ is uniquely represented as a finite sum

$$A = \sum_{k=0}^{n-1} A_k, \tag{10}$$

where $A_k \in \mathbf{A}_k$, it is obvious that the formula $F(A) = A_0$ defines the linear operator on the normed space $P(L)$ satisfying the two conditions mentioned above.

We need to prove the boundedness of the operator F. To this end, we fix an element $A \in P(L)$ and consider its representation of form (10).

Let us show that the following estimation for norms is valid:

$$\|F(A)\| = \|A_0\| \le \|A\|. \tag{11}$$

To do this, we recall that for the Hilbert space $l^2(L)$ there is decomposition (5) into the orthogonal sum of subspaces. By Lemma 2, the subspaces H_k are invariant under the action of $\mathbf{A}_0$. Therefore, for every element $A_0 \in \mathbf{A}_0$ the following decomposition into the direct sum of the restrictions of the operator to the subspaces holds:

$$A_0 = A_0^{(0)} \oplus \ldots \oplus A_0^{(n-1)},$$

where $A_0^{(k)} = A_0\big|_{H_k}$, $0 \le k \le n-1$. Consequently, we have the following equality:

$$\|A_0\| = \max_{0 \le k \le n-1} \left\|A_0^{(k)}\right\|.$$

Let k_0 be an integer such that the equality $\|A_0\| = \left\|A_0^{(k_0)}\right\|$ holds.

Now we note that the following inequality is valid:

$$\|A\| = \sup_{\|x\|=1,\, x \in l^2(L)} (Ax, Ax)^{\frac{1}{2}} \ge \sup_{\|x\|=1,\, x \in H_{k_0}} (Ax, Ax)^{\frac{1}{2}}.$$

Further, we evaluate the inner product (Ax, Ax) for a vector $x \in H_{k_0}$:

$$(Ax, Ax) = \left(\sum_{k=0}^{n-1} A_k x, \sum_{l=0}^{n-1} A_l x \right) = \sum_{k,l=0}^{n-1} (A_k x, A_l x).$$

Let us show that $(A_k x, A_l x) = 0$ whenever $k \neq l$. Indeed, by Lemma 2, if $x \in H_{k_0}$ then $A_k x \in H_r$ and $A_l x \in H_s$, where $[r]_n = [k + k_0]_n$ and $[s]_n = [l + k_0]_n$. If $k \neq l$ then $r \neq s$. Using the orthogonality of the subspaces H_r and H_s, we obtain the desired condition $(A_k x, A_l x) = 0$. Thus we have the following estimate:

$$(Ax, Ax) = \sum_{k=0}^{n-1} (A_k x, A_k x) \geq (A_0 x, A_0 x).$$

It implies inequality (11) as follows:

$$\|A\| \geq \sup_{\|x\|=1, x \in H_{k_0}} (Ax, Ax)^{\frac{1}{2}} \geq \sup_{\|x\|=1, x \in H_{k_0}} (A_0 x, A_0 x)^{\frac{1}{2}} = \left\| A_0^{(k_0)} \right\| = \|A_0\|. \blacktriangleright$$

Lemmas 3 and 4 allow us to formulate the main result of the paper on the topological grading of the semigroup C^*-algebra $C_r^*(L)$.

Theorem 2. *Let $\sigma : L \to \mathbf{Z}_n$ be a surjective homomorphism of semigroups and $\mathbf{A}_k$ be the closed subspace in the semigroup C^*-algebra $C_r^*(L)$ that is generated by operators of form (4) with the σ-index $[k]_n$, where $0 \leq k \leq n-1$. Then the family of the subspaces $\{\mathbf{A}_k \mid 0 \leq k \leq n-1\}$ is a topological $\mathbf{Z}_n$-grading of the C^*-algebra $C_r^*(L)$.*

Finally, the following result shows that the underlying Banach space of the semigroup C^*-algebra $C_r^*(L)$ is decomposed into a direct sum of the finite family of its subspaces $\mathbf{A}_k$.

Theorem 3. *Let $\sigma : L \to \mathbf{Z}_n$ be a surjective homomorphism of semigroups and $\mathbf{A}_k$ be the closed subspace in the semigroup C^*-algebra $C_r^*(L)$ that is generated by operators of form (4) with the σ-index $[k]_n$, where $0 \leq k \leq n-1$. Then we have the following equality for the Banach spaces:*

$$C_r^*(L) = \bigoplus_{k=0}^{n-1} \mathbf{A}_k.$$

◄ By [9, § 19.3], since the family of subspaces $\{\mathbf{A}_k \mid 0 \leq k \leq n-1\}$ is a topological $\mathbf{Z}_n$-grading of the semigroup C^*-algebra $C_r^*(L)$ for every non-negative integer k, $k \leq n-1$, there exists a contractive linear operator $F_k : C_r^*(L) \to \mathbf{A}_k$ satisfying the following condition. For every finite sum

$$A = \sum_{k=0}^{n-1} A_k, \tag{12}$$

where $A_k \in \mathbf{A}_k$, one has the equality $F_k(A) = A_k$.

Let us take an arbitrary element $B \in C_r^*(L)$. We assert that the following equality holds:

$$B = \sum_{k=0}^{n-1} F_k(B). \tag{13}$$

Indeed, let us fix a real number $\varepsilon > 0$. In the semigroup C^*-algebra $C_r^*(L)$ we consider the subalgebra $P(L)$ consisting of the finite linear combinations of monomials. Using the density of this subalgebra, we take a sequence of elements $A_m \in P(L)$, $m \in \mathbf{N}$, such that

$$\lim_{m \to +\infty} A_m = B. \tag{14}$$

Obviously, each element A_m can be represented in form (12). Therefore, by condition (14), the standard arguments allow us to choose a number $m \in \mathbf{N}$ such that the following estimates for the norms are valid:

$$\left\| B - \sum_{k=0}^{n-1} F_k(B) \right\| \le \| B - A_m \| + \left\| \sum_{k=0}^{n-1} F_k(A_m) - \sum_{k=0}^{n-1} F_k(B) \right\| \le (n+1) \| B - A_m \| \le \varepsilon.$$

Since the choice of the number $\varepsilon > 0$ is arbitrary, equality (13) follows, as asserted. This completes the proof of the theorem. ►

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