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GENERALIZATION OF BASS — GURA FORMULA FOR LINEAR DYNAMIC SYSTEMS WITH VECTOR CONTROL

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Abstract

The compact analytic formula of calculating the feedback law (controller matrix) coefficients is developed for solving the synthesis problem of modal controller providing desired pole placement by means of the fully measured state vector in linear dynamic systems with vector control. This formula represents the generalization of the known Bass — Gura formula, used for synthesizing modal controllers in systems with scalar control, to systems with vector control. The obtained solution is applicable to systems with state-space dimension divisible by the number of control inputs and the matrix composed of the linearly independent first block columns of the Kalman controllability matrix by a number corresponding to the quantity of the mentioned multiplicity is reversible. To use the mentioned formula, it's not required to additionally transfer the described systems of the indicated class to special canonical forms. This formula may be applied to solve both numeric and analytic problems of modal control in mentioned class, independently on a specific ratio of state-vector and control-vector dimensions as well as on existence and multiplicity of real-value poles and complex-conjugate pairs of poles in original and desirable spectrums of state matrix. The examples are considered that prove the possibility of applying the generalized block-matrix Bass — Gura formula to calculate modal controllers for the described class of systems with vector control

Keywords

Automatic control system, modal controller, analytic solution, scalar control, vector control, state-vector feedback, matrix spectrum, characteristic polynomial, block-matrix, similarity transformation, block transposition of a matrix

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Introduction and problem statement. One of the modern directions of improving the theory of automatic control is the development of modal synthesis for linear systems with many inputs and many outputs when controlling both according to the state [1–3] and the output in the form of analytical [4, 5] and numerical [6–11] algorithms. The calculation of modal controllers and observers with state-vector feedback for linear systems with scalar control can be carried out based on Bass — Gura formula [12, 13]. The possibility of applying a similar formula to systems with vector control has not been elaborated.

Let us formulate the modal control problem for dynamic systems with scalar control.

A fully controlled k-dimensional ($k \in \mathbb{N}$, $\mathbb{N}$ is a set of natural numbers) by the state ($\mathbf{x} \in \mathbb{R}^k$, $\mathbb{R}^k$ is a set of vectors of dimension $k \times 1$ with real elements) dynamic system with one ($m = 1$) control input ($u \in \mathbb{R}$, $\mathbb{R}$ is a set of real numbers) is specified:

$$\sigma\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \tag{1}$$

where $\sigma\mathbf{x}(t)$ is the operator corresponding to the differentiation operation $\dot{\mathbf{x}}(t)$ for the continuous time case and the shift operation $\mathbf{x}(t+1)$ for the discrete time case; $\mathbf{A} \in \mathbb{R}^{k\times k}$ is the state matrix ($\mathbb{R}^{k\times k}$ is a set of matrices of dimension $k \times k$ with real elements); $\mathbf{b} \in \mathbb{R}^k$ is the control vector. The complete controllability of the system (1) corresponds to the non-singular Kalman controllability matrix

$$\mathbf{\Omega}(\mathbf{A},\mathbf{b}) = \left[\mathbf{A}^0\mathbf{b} \mid \mathbf{A}^1\mathbf{b} \mid \ldots \mid \mathbf{A}^{k-2}\mathbf{b} \mid \mathbf{A}^{k-1}\mathbf{b}\right], \quad \left|\mathbf{\Omega}(\mathbf{A},\mathbf{b})\right| \neq 0. \tag{2}$$

It is required to determine the only possible controller vector $\mathbf{k}^T \in \mathbb{R}^{1\times k}$, that provides the state matrix of the closed-loop system object–controller $\mathbf{A} - \mathbf{b}\mathbf{k}^T$ with a given spectrum (a set of eigenvalues):

$$\operatorname{eig}(\mathbf{A} - \mathbf{b}\mathbf{k}^T) = \Lambda^* = \left\{\lambda_1^*,\ \lambda_2^*,\ \ldots,\ \lambda_{k-1}^*,\ \lambda_k^*\right\}. \tag{3}$$

Here is the theorem [12, 13], on the basis of which the problem under consideration can be solved.

Theorem 1. The Bass — Gura formula for a system with scalar control. *Let in the modal control problem described by relations (1)–(3) be composed: the original characteristic vector* ($p_0,\ p_1,\ \ldots,\ p_{k-1} \in \mathbb{R}$)

$$\mathbf{p}(\mathbf{A},\mathbf{b}) = \left[p_0 \mid p_1 \mid \ldots \mid p_{k-2} \mid p_{k-1}\right]^T = -\mathbf{\Omega}^{-1}(\mathbf{A},\mathbf{b})\mathbf{A}^k\mathbf{b}$$

and the symmetric matrix formed from its components

$$\mathbf{T}_2(\mathbf{A},\mathbf{b}) = \left[\begin{array}{c|c|c|c|c} p_1 & p_2 & \cdots & p_{n-1} & 1 \\ p_2 & \iddots & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ p_{n-1} & 1 & \iddots & \iddots & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{array}\right],$$

as well as the desirable characteristic vector $(p_0^*,\ p_1^*,\ \ldots,\ p_{k-1}^* \in \mathbb{R})$

$$\mathbf{p}^* = \left[\, p_0^* \mid p_1^* \mid \cdots \mid p_{k-2}^* \mid p_{k-1}^* \,\right]^T,$$

such that the set of solutions to the equation

$$\left|\lambda^k + p_{k-1}^*\lambda^{k-1} + \ldots + p_1^*\lambda + p_0^*\right| = 0 \tag{4}$$

with respect to the variable λ *forms a given spectrum* Λ^*. *Then the desired vector of the controller is determined by the Bass — Gura formula [12, 13]*

$$\mathbf{k}^T = \left(\mathbf{p}^{*T} - \mathbf{p}^T(\mathbf{A},\mathbf{b})\right)\left(\mathbf{\Omega}(\mathbf{A},\mathbf{b})\mathbf{T}_2(\mathbf{A},\mathbf{b})\right)^{-1}. \tag{5}$$

Finding the desirable characteristic polynomial (vector $\mathbf{p}^*$). The modal control problem with one control input has a unique solution; therefore, the desirable polynomial from (4) with coefficients $p_0^*,\ p_1^*,\ \ldots,\ p_{k-1}^*$ is determined uniquely from the values of the poles in the spectrum Λ^*. The components $p_0^*,\ p_1^*,\ \ldots,\ p_{k-1}^*$ coincide with the coefficients for the corresponding degrees λ of the polynomial

$$\lambda^k + p_{k-1}^*\lambda^{k-1} + \ldots + p_1^*\lambda + p_0^* = \left(\lambda - \lambda_1^*\right)\left(\lambda - \lambda_2^*\right)\ldots\left(\lambda - \lambda_k^*\right).$$

The paper considers a generalization of the Bass — Gura formula (5) for a certain class of dynamic systems with vector control. This generalization is relevant since currently there are no explicit formulas for calculating controllers for such systems similar to the Bass — Gura formula (5). Let us formulate the modal control problem for the indicated class of dynamic systems with vector control.

A fully controlled n-dimensional $(n \in \mathbb{N})$ by the state $(\mathbf{x} \in \mathbb{R}^n)$ dynamic system with several $(m \in \mathbb{N},\ m > 1)$ control inputs $(\mathbf{u} \in \mathbb{R}^m)$ is specified:

$$\sigma\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{6}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the state matrix; $\mathbf{B} \in \mathbb{R}^{n\times m}$ is the control matrix. Wherein

$$k = \frac{n}{m} \in \mathbb{N}, \tag{7}$$

and the matrix

$$\mathbf{\Omega}(\mathbf{A},\mathbf{B}) = \left[\,\mathbf{A}^0\mathbf{B} \mid \mathbf{A}^1\mathbf{B} \mid \ldots \mid \mathbf{A}^{k-2}\mathbf{B} \mid \mathbf{A}^{k-1}\mathbf{B}\,\right],\quad \left|\mathbf{\Omega}(\mathbf{A},\mathbf{B})\right| \neq 0, \tag{8}$$

composed of the first k block columns, i.e., of the first n columns of the Kalman controllability matrix for a pair of matrices $(\mathbf{A},\ \mathbf{B})$ is invertible.

It is required to determine the set of controller matrices $\mathbf{K} \in \mathbb{R}^{m\times n}$ that provides the state matrix of the closed-loop system object–controller $\mathbf{A}-\mathbf{BK}$ with a given spectrum

$$\operatorname{eig}(\mathbf{A}-\mathbf{BK}) = \Lambda^* = \left\{\lambda_1^*,\ \lambda_2^*,\ \ldots,\ \lambda_{mk-1}^*,\ \lambda_{mk}^*\right\}. \quad (9)$$

To solve this problem with vector control for block matrices and vectors, by analogy with the standard transpose operation, denoted by the superscript "T", it is used the block transposition operation, denoted by the superscript "$\boldsymbol{T}$". As a result of the block transposition operation, square brackets of dimension $m\times m$ move to positions that are symmetrical with respect to the main block diagonal (in the case of a non-square matrix, this diagonal is arbitrary), and inside the blocks themselves, the scalar elements do not move.

Let us formulate a theorem on the basis of which the problem under consideration can be solved.

Theorem 2. Bass — Gura formula for a system with vector control. *Let in the modal control problem described by relations (6)–(9), be composed: the original block-matrix characteristic vector* $(\mathbf{P}_0,\ \mathbf{P}_1,\ \ldots,\ \mathbf{P}_{k-1} \in \mathbb{R}^{m\times m})$

$$\mathbf{P}(\mathbf{A},\mathbf{B}) = \left[\mathbf{P}_0 \mid \mathbf{P}_1 \mid \ldots \mid \mathbf{P}_{k-2} \mid \mathbf{P}_{k-1}\right]^{\boldsymbol{T}} = -\boldsymbol{\Omega}^{-1}(\mathbf{A},\mathbf{B})\mathbf{A}^k\mathbf{B} \quad (10)$$

and the block-symmetric matrix formed from its components

$$\mathbf{T}_2(\mathbf{A},\mathbf{B}) = \left[\begin{array}{c|c|c|c|c} \mathbf{P}_1 & \mathbf{P}_2 & \cdots & \mathbf{P}_{k-1} & \mathbf{I}_m \\ \mathbf{P}_2 & \iddots & \iddots & \mathbf{I}_m & \mathbf{0}_{m\times m} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \mathbf{P}_{k-1} & \mathbf{I}_m & \iddots & \iddots & \mathbf{0}_{m\times m} \\ \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} \end{array}\right], \quad (11)$$

where $\mathbf{I}_m$ *is the identity matrix of dimension* $m\times m$; $\mathbf{0}_{m\times m}$ *is the zero matrix of dimension* $m\times m$, *and also the desirable block-matrix characteristic vector* $(\mathbf{P}_0^*,\ \mathbf{P}_1^*,\ \ldots,\ \mathbf{P}_{k-1}^* \in \mathbb{R}^{m\times m})$:

$$\mathbf{P}^* = \left[\mathbf{P}_0^* \mid \mathbf{P}_1^* \mid \ldots \mid \mathbf{P}_{k-2}^* \mid \mathbf{P}_{k-1}^*\right]^{\boldsymbol{T}}, \quad (12)$$

such that the set of solutions to the equation

$$\left|\lambda^k\mathbf{I}_m + \lambda^{k-1}\mathbf{P}_{k-1}^* + \ldots + \lambda\mathbf{P}_1^* + \mathbf{P}_0^*\right| = 0 \quad (13)$$

with respect to the variable λ *forms a given spectrum* Λ^*. *Then the desired set of controller matrices by the state* $\mathbf{K}$ *can be described by the generalized Bass — Gura block-matrix formula*

$$\mathbf{K} = \left(\mathbf{P}^{*T} - \mathbf{P}^{T}(\mathbf{A},\mathbf{B})\right)\left(\mathbf{\Omega}(\mathbf{A},\mathbf{B})\,\mathrm{T}_2(\mathbf{A},\mathbf{B})\right)^{-1}. \qquad (14)$$

Finding the desirable block-matrix characteristic polynomial (block-matrix vector $\mathbf{P}^*$). The modal control problem with several control inputs does not have a unique solution; therefore, the desirable block-matrix polynomial from (13) with coefficients $\mathbf{P}_0^*,\ \mathbf{P}_1^*,\ \ldots,\ \mathbf{P}_{k-1}^*$ is determined ambiguously from the values of the poles in the spectrum Λ^*. As its blocks, it suffices to take the matrix coefficients for the corresponding degrees λ of the block-matrix polynomial

$$\lambda^k\mathbf{I}_m + \lambda^{k-1}\mathbf{P}_{k-1}^* + \ldots + \lambda\mathbf{P}_1^* + \mathbf{P}_0^* = \left(\lambda\mathbf{I}_m - \mathbf{\Phi}_0\right)\left(\lambda\mathbf{I}_m - \mathbf{\Phi}_1\right)\ldots\left(\lambda\mathbf{I}_m - \mathbf{\Phi}_{k-1}\right), \qquad (15)$$

where $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{k-1} \in \mathbb{C}^{m\times m}$ are such complex matrices of dimension $m \times m$, that their combined spectrum coincides with the given spectrum, i.e., $\operatorname{eig}\mathbf{\Phi}_0 \bigcup \operatorname{eig}\mathbf{\Phi}_1 \bigcup \ldots \bigcup \operatorname{eig}\mathbf{\Phi}_{k-1} = \Lambda^*$, and the blocks $\mathbf{P}_0^*,\ \mathbf{P}_1^*,\ \ldots,\ \mathbf{P}_{k-1}^*$ are obtained by real matrices. This option of assigning matrix coefficients $\mathbf{P}_0^*,\ \mathbf{P}_1^*,\ \ldots,\ \mathbf{P}_{k-1}^* \in \mathbb{R}^{m\times m}$ makes it possible to implement in the solutions of equation (13) any allowable spectrum Λ^* represented by both real numbers and complex conjugate pairs, regardless of their multiplicity and value of the k. It is further shown in numerical and analytical examples.

For matrices with the desired spectra $\mathbf{\Phi}_0,\ \mathbf{\Phi}_1,\ \ldots,\ \mathbf{\Phi}_{k-1}$, one can introduce parameterization based on similarity transformations

$$\mathbf{\Phi}_0 = \mathbf{T}_{\mathbf{\Phi}_0}^{-1}\tilde{\mathbf{\Phi}}_0\mathbf{T}_{\mathbf{\Phi}_0},\ \mathbf{\Phi}_1 = \mathbf{T}_{\mathbf{\Phi}_1}^{-1}\tilde{\mathbf{\Phi}}_1\mathbf{T}_{\mathbf{\Phi}_1}, \ldots, \mathbf{\Phi}_{k-1} = \mathbf{T}_{\mathbf{\Phi}_{k-1}}^{-1}\tilde{\mathbf{\Phi}}_{k-1}\mathbf{T}_{\mathbf{\Phi}_{k-1}},$$

where $\mathbf{T}_{\mathbf{\Phi}_0}, \mathbf{T}_{\mathbf{\Phi}_1}, \ldots, \mathbf{T}_{\mathbf{\Phi}_{k-1}}$ are non-singular transformation matrices; $\tilde{\mathbf{\Phi}}_0, \tilde{\mathbf{\Phi}}_1, \ldots, \tilde{\mathbf{\Phi}}_{k-1}$ are matrices with the same spectra as the corresponding matrices $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{k-1}$, but having a simpler (for example, diagonal) form.

Moreover, we note that in the general case, the desirable block-matrix polynomial from (13) does not have to be resolved into matrix factors (15). The main thing is that the solutions of equation (13) form a given spectrum Λ^*.

Therefore, the desirable block-matrix polynomial, written in the form of vector $\mathbf{P}^*$ (12), provides the parameterization of the desired controller matrix $\mathbf{K}$ both by parameterizing the matrices included in it $\mathbf{\Phi}_0, \mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{k-1}$, and by adding more parameters to its record that do not change the set of solutions of equation (13). This allows, in addition to the desired pole placement, to simultaneously solve other special problems, for example, to minimize the norm of the controller matrix.

Proof of Theorem 2. In considering the class of dynamic systems with vector control indicated in Theorem 2, we assume that the matrices $\mathbf{A}$, $\mathbf{B}$, and also all the matrices introduced below are divided into blocks of dimension $m \times m$, unless a different partition is introduced. Let us denote such blocks by lower indices $[\eta,\ \mu]$. For example, $\mathbf{A}_{[1:k]}$ $(\eta = 1,\ \mu = k)$ is the upper right block of the matrix $\mathbf{A}$.

Based on the derivation of the Bass — Gura formula for systems with scalar control [13], we reduce the considered pair of matrices $(\mathbf{A},\ \mathbf{B})$ from (4) to the block canonical form

$$\tilde{\tilde{\mathbf{A}}} = \left[\begin{array}{c|c|c|c|c} \mathbf{0}_{m\times m} & \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{I}_m & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{I}_m \\ -\mathbf{P}_0 & -\mathbf{P}_1 & \cdots & -\mathbf{P}_{k-2} & -\mathbf{P}_{k-1} \end{array}\right], \quad \tilde{\tilde{\mathbf{B}}} = \begin{bmatrix} \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ \mathbf{I}_m \end{bmatrix}, \tag{16}$$

a similar controlled Luenberger form [13, 14] for the scalar control case. In this form, the state matrix $\tilde{\tilde{\mathbf{A}}}$ is a block analogue of the accompanying matrix of the characteristic polynomial of the original state matrix $\mathbf{A}$. In the bottom (k-th) block row of matrix $\tilde{\tilde{\mathbf{A}}}$ block-matrix coefficients of vector (10) are written. A pair of matrices (16) corresponds to the differential equation written in the Cauchy form

$$\mathbf{q}^{(k)} + \mathbf{P}_{k-1}\mathbf{q}^{(k-1)} + \ldots + \mathbf{P}_1\dot{\mathbf{q}} + \mathbf{P}_0\mathbf{q} = \mathbf{u},$$

where $\mathbf{q}^{(\mu)}$ $(\mu = 0,\ 1,\ \ldots,\ k)$ is the μ-th time derivative of the vector $\mathbf{q} \in \mathbb{R}^m$. The Laplace transformation of this equation allows us to state that the characteristic polynomial of a matrix of the form $\tilde{\tilde{\mathbf{A}}}$ is determined by the equality

$$\operatorname{poly}(\tilde{\tilde{\mathbf{A}}}) = \left|\lambda^k \mathbf{I}_m + \lambda^{k-1}\mathbf{P}_{k-1} + \ldots + \lambda\mathbf{P}_1 + \mathbf{P}_0\right|. \tag{17}$$

Next, we turn our attention to the algorithm for reducing a pair of matrices $(\mathbf{A},\ \mathbf{B})$ to block canonical form (16). By analogy with the Bass — Gura algorithm for systems with scalar control [13], the reduction is carried out in two stages.

The first similarity transformation is performed with a non-singular transformation matrix (8)

$$\mathbf{T}_1 = \mathbf{\Omega}. \tag{18}$$

As a result of such a transformation, taking into account (10), we obtain a pair of matrices

$$\tilde{\mathbf{A}} = \mathbf{T}_1^{-1}\mathbf{A}\mathbf{T}_1 = \left[\begin{array}{c:c:c:c:c} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & -\mathbf{P}_0 \\ \mathbf{I}_m & \mathbf{0}_{m\times m} & \ddots & \vdots & -\mathbf{P}_1 \\ \mathbf{0}_{m\times m} & \ddots & \ddots & \mathbf{0}_{m\times m} & \vdots \\ \vdots & \ddots & \mathbf{I}_m & \mathbf{0}_{m\times m} & -\mathbf{P}_{k-2} \\ \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_m & -\mathbf{P}_{k-1} \end{array}\right],$$

$$\tilde{\mathbf{B}} = \mathbf{T}_1^{-1}\mathbf{B} = \begin{bmatrix} \mathbf{I}_m \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} \end{bmatrix}. \tag{19}$$

Comparing (16) and (19), we note that $\tilde{\mathbf{A}} = \tilde{\tilde{\mathbf{A}}}^T$.

The second similarity transformation is performed with the transformation matrix

$$\mathbf{T}_2 = \tilde{\tilde{\mathbf{\Omega}}}^{-1}, \tag{20}$$

where

$$\tilde{\tilde{\mathbf{\Omega}}} = \left[\begin{array}{c:c:c:c:c} \tilde{\tilde{\mathbf{A}}}^0\tilde{\tilde{\mathbf{B}}} & \tilde{\tilde{\mathbf{A}}}^1\tilde{\tilde{\mathbf{B}}} & \cdots & \tilde{\tilde{\mathbf{A}}}^{k-2}\tilde{\tilde{\mathbf{B}}} & \tilde{\tilde{\mathbf{A}}}^{k-1}\tilde{\tilde{\mathbf{B}}} \end{array}\right] \tag{21}$$

is a matrix formed from the first k block columns, i.e., from the first n columns of the Kalman controllability matrix for a pair $(\tilde{\tilde{\mathbf{A}}},\ \tilde{\tilde{\mathbf{B}}})$.

Let us show that the matrix $\tilde{\tilde{\mathbf{\Omega}}}$ is invertible and define the matrix inverse to it $\mathbf{T}_2$. To do this, we consider successively the block columns of the matrix (21) using the values of $\tilde{\tilde{\mathbf{A}}}$ and $\tilde{\tilde{\mathbf{B}}}$ from (16). The first block column consists of zero blocks in all positions except the extreme bottom position, in which there is a unit block. In each subsequent block column, after multiplying the previous column from the left by the matrix $\tilde{\tilde{\mathbf{A}}}$ all blocks are shifted one position up. At the same time, the block that was in the extreme upper position disappears, and in the vacated extreme lower position, new blocks appear successively from column to column. Thus, the matrix $\tilde{\tilde{\boldsymbol{\Omega}}}$ has a lower block-triangular view with respect to the secondary block diagonal:

$$\tilde{\tilde{\boldsymbol{\Omega}}} = \left[\begin{array}{c|c|c|c|c} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_m \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \iddots & \mathbf{I}_m & \mathbf{R}_1 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \mathbf{0}_{m\times m} & \mathbf{I}_m & \iddots & \iddots & \mathbf{R}_{k-2} \\ \mathbf{I}_m & \mathbf{R}_1 & \cdots & \mathbf{R}_{k-2} & \mathbf{R}_{k-1} \end{array}\right], \tag{22}$$

where

$$\mathbf{R}_\eta = \begin{cases} \mathbf{I}_m, \ \eta = 0; \\ -\sum\limits_{\mu=1}^{\eta} \mathbf{P}_{k-\mu}\mathbf{R}_{\eta-\mu}, \ \eta = 1, \ 2, \ \ldots, \ k-1. \end{cases}$$

Assuming $\mathbf{P}_k = \mathbf{I}_m$, it is convenient to rewrite the last equalities as

$$\sum_{\mu=0}^{\eta} \mathbf{P}_{k-\mu}\mathbf{R}_{\eta-\mu} = \begin{cases} \mathbf{I}_m, \ \eta = 0; \\ \mathbf{0}_{m\times m}, \ \eta = 1, \ 2, \ \ldots, \ k-1. \end{cases} \tag{23}$$

Matrix (22) belongs to the class of recursively defined block matrices

$$\tilde{\tilde{\boldsymbol{\Omega}}}_r = \begin{cases} \mathbf{I}_m, \ r = 0, \\ \left[\begin{array}{c|c} \mathbf{0}_{m\times r\cdot m} & \mathbf{I}_m \\ \tilde{\tilde{\boldsymbol{\Omega}}}_{r-1} & \mathbf{R}_{[1:r,1]} \end{array}\right], \ r \in \mathbb{N}, \end{cases} \quad \mathbf{R}_{[1:r,1]} = \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \vdots \\ \mathbf{R}_r \end{bmatrix}.$$

Block-triangular matrices of the form $\tilde{\tilde{\boldsymbol{\Omega}}}_r$ are invertible, since the block diagonal with respect to which the triangular form is obtained contains only unit blocks. Let us invert these matrices using Frobenius formulas for inverting block matrices [15]:

$$\tilde{\tilde{\boldsymbol{\Omega}}}_r^{-1} = \begin{cases} \mathbf{I}_m, \ r = 0; \\ \left[\begin{array}{c|c} \mathbf{0}_{rm\times m} & \mathbf{I}_{rm} \\ \mathbf{I}_m & \mathbf{0}_{m\times rm} \end{array}\right] \left[\begin{array}{c|c} \mathbf{I}_m & \mathbf{0}_{m\times r\cdot m} \\ \mathbf{R}_{[1:r,1]} & \tilde{\tilde{\boldsymbol{\Omega}}}_{r-1} \end{array}\right]^{-1} = \left[\begin{array}{c|c} -\tilde{\tilde{\boldsymbol{\Omega}}}_{r-1}^{-1}\mathbf{R}_{[1:r,1]} & \tilde{\tilde{\boldsymbol{\Omega}}}_{r-1}^{-1} \\ \mathbf{I}_m & \mathbf{0}_{m\times rm} \end{array}\right], \ r \in \mathbb{N}. \end{cases}$$

Hence if $r = k-1$ by recursion, we find the matrix inverse to (22):

$$\tilde{\tilde{\boldsymbol{\Omega}}}^{-1}=\left[\begin{array}{c:c:c:c:c}\mathbf{Q}_{k-1} & \mathbf{Q}_{k-2} & \cdots & \mathbf{Q}_1 & \mathbf{I}_m\\ \mathbf{Q}_{k-2} & \iddots & \iddots & \mathbf{I}_m & \mathbf{0}_{m\times m}\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ \mathbf{Q}_1 & \mathbf{I}_m & \iddots & \iddots & \mathbf{0}_{m\times m}\\ \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m}\end{array}\right], \tag{24}$$

where

$$\mathbf{Q}_\eta=\begin{cases}\mathbf{I}_m, \ \eta=0;\\ -\sum\limits_{\mu=0}^{\eta-1}\mathbf{Q}_\mu\mathbf{R}_{\eta-\mu}, \ \eta=1,\ 2,\ \ldots,\ k-1.\end{cases}$$

Since $\mathbf{R}_0=\mathbf{I}_m$ the last equalities can be rewritten as

$$\sum_{\mu=0}^{\eta}\mathbf{Q}_\mu\mathbf{R}_{\eta-\mu}=\begin{cases}\mathbf{I}_m, \ \eta=0;\\ \mathbf{0}_{m\times m}, \ \eta=1,\ 2,\ \ldots,\ k-1.\end{cases} \tag{25}$$

Comparing formulas (23) and (25), successively for values $\eta=0,\ 1,\ \ldots,\ k-1,$ we could conclude that $\mathbf{Q}_\eta=\mathbf{P}_{k-\eta}$. Thus, formula (11) follows from (20) and (24).

Further, let us show that using the second similarity transformation with the transformation matrix $\mathbf{T}_2$ (11) the pair of matrices $(\tilde{\mathbf{A}},\ \tilde{\mathbf{B}})$ (19) becomes the canonical form $(\tilde{\tilde{\mathbf{A}}},\ \tilde{\tilde{\mathbf{B}}})$ (16).

The transformed state matrix, taking into account (11), (19) and (22), is equal to

$$\mathbf{T}_2^{-1}\tilde{\mathbf{A}}\mathbf{T}_2=\tilde{\tilde{\boldsymbol{\Omega}}}\left(\tilde{\mathbf{A}}\mathbf{T}_2\right)=$$

$$=\left[\begin{array}{c:c:c:c:c}\mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_m\\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \iddots & \mathbf{I}_m & \mathbf{R}_1\\ \vdots & \iddots & \iddots & \iddots & \vdots\\ \mathbf{0}_{m\times m} & \mathbf{I}_m & \iddots & \iddots & \mathbf{R}_{k-2}\\ \mathbf{I}_m & \mathbf{R}_1 & \cdots & \mathbf{R}_{k-2} & \mathbf{R}_{k-1}\end{array}\right]\left[\begin{array}{c:c:c:c:c}-\mathbf{P}_0 & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m}\\ \mathbf{0}_{m\times m} & \mathbf{P}_2 & \cdots & \mathbf{P}_{k-1} & \mathbf{I}_m\\ \vdots & \vdots & \iddots & \mathbf{I}_m & \mathbf{0}_{m\times m}\\ \mathbf{0}_{m\times m} & \mathbf{P}_{k-1} & \iddots & \iddots & \vdots\\ \mathbf{0}_{m\times m} & \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m}\end{array}\right]=$$

$$=\left[\begin{array}{c:c:c:c:c}\mathbf{0}_{m\times m} & \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m}\\ \mathbf{0}_{m\times m} & \sum\limits_{\mu=0}^{1}\mathbf{R}_{1-\mu}\mathbf{P}_{k-\mu} & \mathbf{I}_m & \ddots & \vdots\\ \vdots & \vdots & \ddots & \ddots & \mathbf{0}_{m\times m}\\ \mathbf{0}_{m\times m} & \sum\limits_{\mu=0}^{k-2}\mathbf{R}_{(k-2)-\mu}\mathbf{P}_{k-\mu} & \cdots & \sum\limits_{\mu=0}^{1}\mathbf{R}_{1-\mu}\mathbf{P}_{k-\mu} & \mathbf{I}_m\\ -\mathbf{P}_0 & -\mathbf{P}_1+\sum\limits_{\mu=0}^{k-1}\mathbf{R}_{(k-1)-\mu}\mathbf{P}_{k-\mu} & \cdots & -\mathbf{P}_{k-2}+\sum\limits_{\mu=0}^{2}\mathbf{R}_{2-\mu}\mathbf{P}_{k-\mu} & -\mathbf{P}_{k-1}+\sum\limits_{\mu=0}^{1}\mathbf{R}_{1-\mu}\mathbf{P}_{k-\mu}\end{array}\right].$$

Here there are sums that differ from the sums that are written in (23), only in the order of the factors $\mathbf{P}_{k-\mu}$ and $\mathbf{R}_{\eta-\mu}$ in each term. However, these sums, like the corresponding sums in (23), are zero, since the matrix (22) is block-symmetric. Indeed, then in (21)

$$\tilde{\tilde{\boldsymbol{\Omega}}} = \tilde{\tilde{\boldsymbol{\Omega}}}^{\boldsymbol{T}} = \begin{bmatrix} \left[\mathbf{0}_{m\times m} \mid \mathbf{0}_{m\times m} \mid \cdots \mid \mathbf{0}_{m\times m} \mid \mathbf{I}_m\right]\tilde{\mathbf{A}}^0 \\ \left[\mathbf{0}_{m\times m} \mid \mathbf{0}_{m\times m} \mid \cdots \mid \mathbf{0}_{m\times m} \mid \mathbf{I}_m\right]\tilde{\mathbf{A}}^1 \\ \vdots \\ \left[\mathbf{0}_{m\times m} \mid \mathbf{0}_{m\times m} \mid \cdots \mid \mathbf{0}_{m\times m} \mid \mathbf{I}_m\right]\tilde{\mathbf{A}}^{k-2} \\ \left[\mathbf{0}_{m\times m} \mid \mathbf{0}_{m\times m} \mid \cdots \mid \mathbf{0}_{m\times m} \mid \mathbf{I}_m\right]\tilde{\mathbf{A}}^{k-1} \end{bmatrix}, \qquad (26)$$

since $\tilde{\tilde{\mathbf{A}}} = \tilde{\mathbf{A}}^T$. Let us consider successively the block rows of the matrix (26) using the value $\tilde{\mathbf{A}}$ from (19). The first block row consists of zero blocks in all positions except the extreme right position, in which there is a unit block. In each next block row, after multiplying the previous row from the right by the matrix $\tilde{\mathbf{A}}$ all blocks are shifted by one position to the left. In this case, the block that was in the extreme left position disappears, and in the vacated extreme right position, new blocks appear successively from row to row. Thus, matrix (26) has the form (22), where

$$\mathbf{R}_\eta = \begin{cases} \mathbf{I}_m, \ \eta = 0; \\ -\sum\limits_{\mu=1}^{\eta} \mathbf{R}_{\eta-\mu}\mathbf{P}_{k-\mu}, \ \eta = 1,\ 2,\ \ldots,\ k-1. \end{cases}$$

Since $\mathbf{P}_k = \mathbf{I}_m$ the last equalities can be rewritten as

$$\sum_{\mu=0}^{\eta} \mathbf{R}_{\eta-j}\mathbf{P}_{k-\mu} = \begin{cases} \mathbf{I}_m, \ \eta = 0, \\ \mathbf{0}_{m\times m}, \ \eta = 1,\ 2,\ \ldots,\ k-1, \end{cases} \qquad (27)$$

So, the transformed state matrix $\mathbf{T}_2^{-1}\tilde{\mathbf{A}}\mathbf{T}_2$ coincides with the state matrix $\tilde{\tilde{\mathbf{A}}}$ in canonical form (16).

The transformed control matrix is

$$\mathbf{T}_2^{-1}\tilde{\mathbf{B}} = \tilde{\tilde{\boldsymbol{\Omega}}}\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_m \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \iddots & \mathbf{I}_m & \mathbf{R}_1 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \mathbf{0}_{m\times m} & \mathbf{I}_m & \iddots & \iddots & \mathbf{R}_{k-2} \\ \mathbf{I}_m & \mathbf{R}_1 & \cdots & \mathbf{R}_{k-2} & \mathbf{R}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{I}_m \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} \\ \vdots \\ \mathbf{0}_{m\times m} \\ \mathbf{I}_m \end{bmatrix},$$

i.e., it coincides with the control matrix $\tilde{\tilde{\mathbf{B}}}$ in canonical form (16).

The matrix of the complete transition from a pair $(\mathbf{A},\ \mathbf{B})$ to a canonical form $(\tilde{\tilde{\mathbf{A}}},\ \tilde{\tilde{\mathbf{B}}})$ is equal to the product of the matrices (18) and (20), i.e., (8) and (11) of two successive similarity transformations:

$$\mathbf{T}(\mathbf{A},\mathbf{B}) = \mathbf{T}_1(\mathbf{A},\mathbf{B})\mathbf{T}_2(\mathbf{A},\mathbf{B}) = \mathbf{\Omega}(\mathbf{A},\mathbf{B})\tilde{\tilde{\mathbf{\Omega}}}^{-1}(\mathbf{A},\mathbf{B}). \qquad (28)$$

We proceed to the solution of the modal control problem, i.e., finding the controller matrix by the state $\tilde{\tilde{\mathbf{K}}}$ for a pair of matrices $(\tilde{\tilde{\mathbf{A}}},\ \tilde{\tilde{\mathbf{B}}})$ in canonical form (16). The state matrix of a closed-loop system is

$$\tilde{\tilde{\mathbf{A}}} - \tilde{\tilde{\mathbf{B}}}\tilde{\tilde{\mathbf{K}}} = \left[\begin{array}{c|c|c|c|c} \mathbf{0}_{m\times m} & \mathbf{I}_m & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{I}_m & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{0}_{m\times m} & \mathbf{I}_m \\ -\tilde{\tilde{\mathbf{K}}}_{[1,1]} - \mathbf{P}_0 & -\tilde{\tilde{\mathbf{K}}}_{[1,2]} - \mathbf{P}_1 & \cdots & -\tilde{\tilde{\mathbf{K}}}_{[1,k-1]} - \mathbf{P}_{k-2} & -\tilde{\tilde{\mathbf{K}}}_{[1,k]} - \mathbf{P}_{k-1} \end{array}\right].$$

It has the same structure as the state matrix $\tilde{\tilde{\mathbf{A}}}$. Therefore, by analogy with (17) the characteristic polynomial of the matrix of a closed system has the form

$$\operatorname{poly}(\tilde{\tilde{\mathbf{A}}} - \tilde{\tilde{\mathbf{B}}}\tilde{\tilde{\mathbf{K}}}) =$$
$$= \left|\lambda^k \mathbf{I}_m + \lambda^{k-1}\left(\tilde{\tilde{\mathbf{K}}}_{[1,k]} + \mathbf{P}_{k-1}\right) + \ldots + \lambda\left(\tilde{\tilde{\mathbf{K}}}_{[1,2]} + \mathbf{P}_1\right) + \left(\tilde{\tilde{\mathbf{K}}}_{[1,1]} + \mathbf{P}_0\right)\right|. \qquad (29)$$

This polynomial must coincide with the given polynomial (13). Comparing (13) and (29), we find the controller matrix for the transformed system, i.e., for a pair of matrices in canonical form (16):

$$\tilde{\tilde{\mathbf{K}}} = \mathbf{P}^{*\boldsymbol{T}} - \mathbf{P}^{\boldsymbol{T}}.$$

The controller matrix for the original system (6) is determined based on the inverse similarity transformation with the transformation matrix (28)

$$\mathbf{K} = \tilde{\tilde{\mathbf{K}}}\mathbf{T}^{-1} = \left(\mathbf{P}^{*\boldsymbol{T}} - \mathbf{P}^{\boldsymbol{T}}\right)\left(\mathbf{\Omega}(\mathbf{A},\mathbf{B})\mathbf{T}_2(\mathbf{A},\mathbf{B})\right)^{-1},$$

which coincides with the formula (14). Theorem 2 is proven.

In the practical application of the proved formula (14), it is recommended to carry out calculations in the following order.

First, (within the framework of the problem statement), the desirable block-matrix characteristic polynomial is formed in the form of the vector $\mathbf{P}^{*\boldsymbol{T}}$ from

(12) using the given spectrum (9) by decomposing into matrix factors (15), or in another special way in order to simultaneously solve additional problems (minimizing the norm controller matrices, zeroing some of its columns, etc.).

Next, the condition (7) is checked and, when it is satisfied, matrix $\mathbf{\Omega}$ (8) is written. If it is reversible, the inverse matrix is calculated $\mathbf{\Omega}^{-1}$. Under conditions (7) and (8) of Theorem 2, the controller matrix $\mathbf{K}$ can be calculated using the generalized block-matrix Bass — Gura formula (14) in accordance with the algorithm below.

From (10) is calculated the block-matrix vector $\mathbf{P}^T$ of coefficients of the original block-matrix characteristic polynomial of the state matrix $\mathbf{A}$.

The matrix of the second similarity transformation is written $\mathbf{T}_2 = \tilde{\tilde{\mathbf{\Omega}}}^{-1}$ from (11) and its inverse matrix $\tilde{\tilde{\mathbf{\Omega}}} = \mathbf{T}_2^{-1}$ from (22).

The desired controller matrix $\mathbf{K}$ is determined by the formula (14). By virtue of the fact that the inverse matrix was previously found $\mathbf{\Omega}^{-1}$, in order to avoid the inversion of a more complex matrix, it is advisable to use the equality $\left(\mathbf{\Omega}\mathbf{T}_2\right)^{-1} = \mathbf{T}_2^{-1}\mathbf{\Omega}^{-1}$.

The check is made to obtain a given spectrum $\mathrm{eig}(\mathbf{A} - \mathbf{BK}) = \Lambda^*$ by forming and solving a characteristic equation $\left|\lambda \mathbf{I}_n - \left(\mathbf{A} - \mathbf{BK}\right)\right| = 0$ with respect to a variable λ.

Examples. Let us consider the application of the generalized Bass — Gura block-matrix formula (14) using numerical and analytical examples.

Numerical example. Let there be a linear stationary dynamic object with a pair of state $\mathbf{A}$ and control matrices $\mathbf{B}$, whose elements are formed using a random number generator:

$$\mathbf{A} = \left[\begin{array}{c:c:c:c:c:c} 0.7784 & 0.6151 & 1.1796 & 0.0661 & -1.0864 & -2.1154 \\ -0.2465 & 0.2795 & 0.8946 & -0.6379 & 0.2403 & 0.6817 \\ -0.9035 & 0.8189 & -0.1429 & -0.8193 & -1.0462 & 0.0087 \\ -0.4959 & 2.0475 & -0.5935 & 0.9680 & 0.6187 & 0.3343 \\ 0.3745 & -0.3237 & 0.2489 & 0.0303 & 1.3050 & -0.5474 \\ -2.3705 & -0.9805 & -1.1298 & 1.3511 & 1.0235 & -1.6510 \end{array}\right],$$

$$\mathbf{B} = \left[\begin{array}{c:c} 0.8928 & -1.3187 \\ -0.5071 & -0.4741 \\ -0.3298 & 0.9874 \\ 0.1684 & -2.0730 \\ 2.5367 & 1.2440 \\ 2.2936 & -0.3152 \end{array}\right].$$

It is required to calculate a controller with a matrix $\mathbf{K}$, that provides the matrix of a closed-loop system object–controller $\mathbf{A}-\mathbf{BK}$ with a given spectrum $\Lambda^* = \{-1 \pm 2i,\ -3,\ -3,\ -3,\ -3\}$, where i is the imaginary unit and the desirable block-matrix characteristic polynomial calculated according to (15) in the form (12):

$$\mathbf{P}^{*T} = \left[\mathbf{P}_0^* \mid \mathbf{P}_1^* \mid \mathbf{P}_2^*\right] =$$
$$= \left[-\mathbf{\Phi}_0\mathbf{\Phi}_1\mathbf{\Phi}_2 \mid \mathbf{\Phi}_0\mathbf{\Phi}_1 + \mathbf{\Phi}_1\mathbf{\Phi}_2 + \mathbf{\Phi}_0\mathbf{\Phi}_2 \mid -\mathbf{\Phi}_0 - \mathbf{\Phi}_1 - \mathbf{\Phi}_2\right] =$$
$$= \begin{bmatrix} 9 & -18 & 15 & -12 & 7 & -2 \\ 18 & 9 & 12 & 15 & 2 & 7 \end{bmatrix}.$$

Here

$$\mathbf{\Phi}_0 = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}, \quad \mathrm{eig}(\mathbf{\Phi}_0) = \{-1 \pm 2i\};$$
$$\mathbf{\Phi}_1 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \quad \mathrm{eig}(\mathbf{\Phi}_1) = \{-3,\ -3\};$$
$$\mathbf{\Phi}_2 = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \quad \mathrm{eig}(\mathbf{\Phi}_2) = \{-3,\ -3\},$$

i.e., $\mathrm{eig}\,\mathbf{\Phi}_0 \cup \mathrm{eig}\,\mathbf{\Phi}_1 \cup \mathrm{eig}\,\mathbf{\Phi}_2 = \Lambda^*$.

The solution of a numerical example. In the problem under consideration, the dimensions of the state $(n = 6)$ and control vectors $(m = 2)$ are such that their ratio $k = n/m = 3$ is an integer. A pair of matrices $(\mathbf{A},\ \mathbf{B})$ is completely controllable according to the Kalman criterion, moreover, matrix (8)

$$\mathbf{\Omega} = \left[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}\right] =$$

$$= \begin{bmatrix} 0.8928 & -1.3187 & -7.6026 & -0.9750 & -7.6429 & -3.0204 \\ -0.5071 & -0.4741 & 1.4090 & 2.4822 & -2.9488 & 4.7057 \\ -0.3298 & 0.9874 & -3.9466 & 1.0563 & 4.9823 & 2.9005 \\ 0.1684 & -2.0730 & 1.2140 & -2.2451 & 10.9659 & 4.2704 \\ 2.5367 & 1.2440 & 2.4765 & 1.6386 & 0.1927 & 0.3608 \\ 2.2936 & -0.3152 & -2.2096 & 1.4682 & 28.9222 & -5.0963 \end{bmatrix},$$

composed of the first three $(k = 3)$ block columns, i.e., the first six $(n = 6)$ columns of the Kalman controllability matrix, has an inverse matrix

$$\mathbf{\Omega}^{-1} = \begin{bmatrix} 0.1354 & -0.0460 & -0.0221 & 0.1184 & 0.3377 & -0.0122 \\ -0.1563 & -0.2959 & 0.2889 & -0.0939 & 0.1178 & -0.0864 \\ -0.0709 & 0.0100 & -0.1013 & 0.0041 & 0.0153 & -0.0019 \\ 0.0189 & 0.2767 & -0.0554 & -0.1374 & -0.0354 & 0.0951 \\ -0.0192 & -0.0088 & 0.0171 & 0.0113 & -0.0130 & 0.0215 \\ -0.0019 & 0.0233 & 0.0970 & 0.0816 & 0.0542 & -0.0461 \end{bmatrix}.$$

Thus, all the conditions of Theorem 2 are satisfied and the controller matrix $\mathbf{K}$ can be calculated using the generalized Bass — Gura block-matrix formula (14).

We calculate from (10) the block-matrix vector of coefficients of the original block-matrix characteristic polynomial of the state matrix $\mathbf{A}$:

$$\mathbf{P}^{\boldsymbol{T}} = \left[\mathbf{P}_0 \;\vdots\; \mathbf{P}_1 \;\vdots\; \mathbf{P}_2 \right] = -\left(\boldsymbol{\Omega}^{-1} \mathbf{A}^3 \mathbf{B} \right)^{\boldsymbol{T}} =$$

$$= \left[\begin{array}{c:c:c:c:c:c} 12.5604 & -3.7503 & -4.9936 & 1.2357 & -0.9547 & -0.1848 \\ -1.0763 & 3.3623 & -0.9861 & 0.6461 & -1.2236 & -0.5822 \end{array} \right].$$

We write the matrix of the second similarity transformation $\mathbf{T}_2 = \tilde{\tilde{\boldsymbol{\Omega}}}^{-1}$ from (11)

$$\mathbf{T}_2 = \left[\begin{array}{c:c:c} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{I}_2 \\ \mathbf{P}_2 & \mathbf{I}_2 & \mathbf{0}_{2\times2} \\ \mathbf{I}_2 & \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} \end{array} \right] = \left[\begin{array}{r:r:r:r:r:r} -4.9936 & 1.2357 & -0.9547 & -0.1848 & 1 & 0 \\ -0.9861 & 0.6461 & -1.2236 & -0.5822 & 0 & 1 \\ -0.9547 & -0.1848 & 1 & 0 & 0 & 0 \\ -1.2236 & -0.5822 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array} \right]$$

and its inverse matrix $\tilde{\tilde{\boldsymbol{\Omega}}} = \mathbf{T}_2^{-1}$ from (22)

$$\mathbf{T}_2^{-1} = \left[\begin{array}{c:c:c} \mathbf{0}_{2\times2} & \mathbf{0}_{2\times2} & \mathbf{I}_2 \\ \mathbf{0}_{2\times2} & \mathbf{I}_2 & -\mathbf{P}_2 \\ \mathbf{I}_2 & -\mathbf{P}_2 & \mathbf{P}_2^2 - \mathbf{P}_1 \end{array} \right] = \left[\begin{array}{r:r:r:r:r:r} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0.9547 & 0.1848 \\ 0 & 0 & 0 & 1 & 1.2236 & 0.5822 \\ 1 & 0 & 0.9547 & 0.1848 & 6.1313 & -0.9517 \\ 0 & 1 & 1.2236 & 0.5822 & 2.8667 & -0.0810 \end{array} \right].$$

We determine the desired controller matrix $\mathbf{K}$ by the formula (14)

$$\mathbf{K} = \left(\mathbf{P}^{*\boldsymbol{T}} - \mathbf{P}^{\boldsymbol{T}} \right) \mathbf{T}_2^{-1} \boldsymbol{\Omega}^{-1} =$$

$$= \left[\begin{array}{r:r:r:r:r:r} -1.4543 & -4.0985 & -4.3175 & 1.3634 & 1.3638 & 0.8989 \\ -3.9384 & 2.7229 & 1.2575 & -0.8313 & 1.1975 & 2.4579 \end{array} \right].$$

We perform a check. We form the characteristic matrix of the closed-loop system object–controller

$$\mathbf{Z} = \lambda \mathbf{I}_6 - \left(\mathbf{A} - \mathbf{B}\mathbf{K} \right) =$$

$$= \left[\begin{array}{r:r:r:r:r:r} \lambda + 3.1167 & -7.8648 & -6.6925 & 2.2473 & 0.7249 & -0.3231 \\ 2.8512 & \lambda + 0.5077 & 0.6984 & 0.3407 & -1.4996 & -2.3029 \\ -2.5056 & 3.2213 & \lambda + 2.8085 & -0.4511 & 1.7788 & 2.1217 \\ 8.4155 & -8.3822 & -2.7403 & \lambda + 0.9848 & -2.8715 & -5.2782 \\ -8.9631 & -6.6857 & -9.6369 & 2.3941 & \lambda + 3.6444 & 5.8855 \\ 0.2765 & -9.2783 & -9.1694 & 2.0381 & 1.7271 & \lambda + 2.9380 \end{array} \right]$$

and we find its determinant

$$|\mathbf{Z}| = \lambda^6 + 14\lambda^5 + 83\lambda^4 + 276\lambda^3 + 567\lambda^2 + 702\lambda + 405 = \\ = \left(\lambda^2 + 2\lambda + 5\right)\left(\lambda + 3\right)^4 .$$

Thus, $\operatorname{eig}(\mathbf{A} - \mathbf{BK}) = \{-1 \pm 2i,\ -3,\ -3,\ -3,\ -3\} = \Lambda^*$, and the calculated controller by the state with the matrix $\mathbf{K}$ solves the posed problem of poles placement.

An analytical example. A linear stationary dynamic object is given with a pair of state $\mathbf{A}$ and control matrices $\mathbf{B}$, containing only real elements:

$$\mathbf{A} = \left[\begin{array}{c:c:c:c:c:c:c:c:c} 0 & a_{12} & 0 & 0 & 0 & 0 & a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & a_{35} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{62} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad \mathbf{B} = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \\ b_{41} & 0 & 0 \\ 0 & b_{52} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

It is required to calculate a controller with a matrix $\mathbf{K}$, that provides the matrix of the closed-loop system object–controller $\mathbf{A} - \mathbf{BK}$ with a given spectrum

$$\Lambda^* = \left\{ \phi_1, \begin{array}{l} \phi_2 = s_{23} - m_{23}i, \ \ \phi_4 = s_{45} - m_{45}i, \ \ \phi_6 = s_{67} - m_{67}i, \ \ \phi_8 = s_{89} - m_{89}i \\ \phi_3 = s_{23} + m_{23}i, \ \ \phi_5 = s_{45} + m_{45}i, \ \ \phi_7 = s_{67} + m_{67}i, \ \ \phi_9 = s_{89} + m_{89}i \end{array} \right\},$$

where $\phi_1,\ s_{23},\ m_{23},\ s_{45},\ m_{45},\ s_{67},\ m_{67},\ s_{89},\ m_{89} \in \mathbb{R}$.

In the considered problem, in addition to the desired pole placement, it is proposed, using the parameterization of the desirable block-matrix characteristic polynomial, written in the form of vector $\mathbf{P}^*$ (12), to further reduce the norm of the controller matrix $\mathbf{K}$ (the sum of squares of its coefficients). For this reason, let us calculate the block-matrix vector (12) based on the expansion (15)

$$\left[-\mathbf{\Phi}_0\mathbf{\Phi}_1\mathbf{\Phi}_2 \;\middle|\; \mathbf{\Phi}_0\mathbf{\Phi}_1 + \mathbf{\Phi}_1\mathbf{\Phi}_2 + \mathbf{\Phi}_0\mathbf{\Phi}_2 \;\middle|\; -\mathbf{\Phi}_0 - \mathbf{\Phi}_1 - \mathbf{\Phi}_2 \right],$$

where parameterized matrices with the desired spectra (due to similarity transformations with real matrices $\mathbf{T}_{\mathbf{\Phi}_0}$, $\mathbf{T}_{\mathbf{\Phi}_1}$ and $\mathbf{T}_{\mathbf{\Phi}_2}$) are introduced

$$\mathbf{\Phi}_0 = \mathbf{T}_{\mathbf{\Phi}_0} \underbrace{\left[\begin{array}{c|c|c} \phi_1 & 0 & 0 \\ 0 & s_{45} & m_{45} \\ 0 & -m_{45} & s_{45} \end{array}\right]}_{\tilde{\mathbf{\Phi}}_0} \mathbf{T}_{\mathbf{\Phi}_0}^{-1}, \ \operatorname{eig}(\mathbf{\Phi}_0) = \left\{ \phi_1, \quad \begin{array}{l} \phi_4 = s_{45} - m_{45} i \\ \phi_5 = s_{45} + m_{45} i \end{array} \right\},$$

$$\mathbf{\Phi}_1 = \mathbf{T}_{\mathbf{\Phi}_1} \underbrace{\left[\begin{array}{c|c|c} s_{23} - m_{23} i & 0 & 0 \\ 0 & s_{67} & m_{67} \\ 0 & -m_{67} & s_{67} \end{array}\right]}_{\tilde{\mathbf{\Phi}}_1} \mathbf{T}_{\mathbf{\Phi}_1}^{-1},$$

$$\operatorname{eig}(\mathbf{\Phi}_1) = \left\{ \phi_2 = s_{23} - m_{23} i, \quad \begin{array}{l} \phi_6 = s_{67} - m_{67} i \\ \phi_7 = s_{67} + m_{67} i \end{array} \right\},$$

$$\mathbf{\Phi}_2 = \mathbf{T}_{\mathbf{\Phi}_2} \underbrace{\left[\begin{array}{c|c|c} s_{23} + m_{23} i & 0 & 0 \\ 0 & s_{89} & m_{89} \\ 0 & -m_{89} & s_{89} \end{array}\right]}_{\tilde{\mathbf{\Phi}}_2} \mathbf{T}_{\mathbf{\Phi}_2}^{-1},$$

$$\operatorname{eig}(\mathbf{\Phi}_2) = \left\{ \phi_3 = s_{23} + m_{23} i, \quad \begin{array}{l} \phi_8 = s_{89} - m_{89} i \\ \phi_9 = s_{89} + m_{89} i \end{array} \right\}.$$

To simplify this illustrative example, the transformation matrices are set equal and contain one variable parameter $\kappa \in \mathbb{R}$ $(\kappa \neq 0)$:

$$\mathbf{T}_{\mathbf{\Phi}_0} = \mathbf{T}_{\mathbf{\Phi}_1} = \mathbf{T}_{\mathbf{\Phi}_2} = \left[\begin{array}{c|c|c} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Moreover, when writing the block-matrix vector $\mathbf{P}^*$ (12) we introduce an additional parameter $c \in \mathbb{R}$, located in a position that does not affect the set of solutions of the characteristic equation (13). Let it be given

$$\mathbf{P}^{*T} = \left[\mathbf{P}_0^* \mid \mathbf{P}_1^* \mid \mathbf{P}_2^* \right] =$$

$$= \left[\begin{array}{c|c|c|c|c|c|c|c|c} f_3 & 0 & 0 & f_2 & 0 & c & f_1 & 0 & 0 \\ 0 & s_3 & \kappa m_3 & 0 & s_2 & \kappa m_2 & 0 & s_1 & \kappa m_1 \\ 0 & -m_3/\kappa & s_3 & 0 & -m_2/\kappa & s_2 & 0 & -m_1/\kappa & s_1 \end{array}\right].$$

Here, for brevity, the following notation is introduced:

$$f_1 = -\phi_1 - \phi_2 - \phi_3 \in \mathbb{R},\quad f_2 = \phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1 \in \mathbb{R},\quad f_3 = -\phi_1\phi_2\phi_3 \in \mathbb{R},$$

$$s_1 = \frac{g_1 + h_1}{2} \in \mathbb{R},\ s_2 = \frac{g_2 + h_2}{2} \in \mathbb{R},\ s_3 = \frac{g_3 + h_3}{2} \in \mathbb{R},$$

$$m_1 = \frac{g_1 - h_1}{2} i \in \mathbb{R},\ m_2 = \frac{g_2 - h_2}{2} i \in \mathbb{R},\ m_3 = \frac{g_3 - h_3}{2} i \in \mathbb{R},$$

$$g_1 = -\phi_4 - \phi_6 - \phi_8 = s_1 - m_1 i,\ g_2 = \phi_4\phi_6 + \phi_6\phi_8 + \phi_8\phi_4 = s_2 - m_2 i,$$

$$g_3 = -\phi_4\phi_6\phi_8 = s_3 - m_3 i,$$

$$h_1 = -\phi_5 - \phi_7 - \phi_9 = s_1 + m_1 i,\ h_2 = \phi_5\phi_7 + \phi_7\phi_9 + \phi_9\phi_5 = s_2 + m_2 i,$$

$$h_3 = -\phi_5\phi_7\phi_9 = s_3 + m_3 i.$$

With such a vector $\mathbf{P}^*$ the left side of equation (13) takes the form

$$\left|\lambda^3 I_m + \lambda^2 P_2^* + \lambda P_1^* + P_0^*\right| = \begin{vmatrix} f_\lambda & 0 & c\lambda \\ 0 & s_\lambda & \kappa m_\lambda \\ 0 & -m_\lambda/\kappa & s_\lambda \end{vmatrix} = f_\lambda \left(s_\lambda^2 + m_\lambda^2\right) = \prod_{\eta=1}^{9} \left(\lambda - \phi_\eta\right),$$

where

$$f_\lambda = \lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 = \left(\lambda - \phi_1\right)\left(\lambda - \phi_2\right)\left(\lambda - \phi_3\right),$$

$$s_\lambda = \lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = \frac{g_\lambda + h_\lambda}{2},\quad m_\lambda = m_1\lambda^2 + m_2\lambda + m_3 = \frac{g_\lambda - h_\lambda}{2} i,$$

$$g_\lambda = \lambda^3 + g_1\lambda^2 + g_2\lambda + g_3 = \left(\lambda - \phi_4\right)\left(\lambda - \phi_6\right)\left(\lambda - \phi_8\right) = s_\lambda - m_\lambda i,$$

$$h_\lambda = \lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = \left(\lambda - \phi_5\right)\left(\lambda - \phi_7\right)\left(\lambda - \phi_9\right) = s_\lambda + m_\lambda i.$$

Therefore, regardless of the parameter values κ and c the set of solutions of equation (13) with respect to λ forms a given spectrum Λ^*.

The solution of an analytical example. In the problem under consideration, the dimensions of the state $(n = 9)$ and control $(m = 3)$ vectors are such that their ratio $k = n / m = 3$ is an integer. A pair of matrices $(\mathbf{A},\ \mathbf{B})$ is completely controllable according to the Kalman criterion, moreover, matrix (8)

$$\mathbf{\Omega} = \left[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}\right] =$$

$$= \left[\begin{array}{c|c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & 0 & a_{12}b_{23} & a_{17}b_{41} & 0 & 0 \\ 0 & 0 & b_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{35}b_{52} & 0 & a_{35}a_{54}b_{41} & a_{33}a_{35}b_{52} & 0 \\ b_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{52} & 0 & a_{54}b_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{62}b_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{52} & 0 & a_{54}b_{41} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{62}b_{23} \end{array}\right]$$

composed of the first three $(k = 3)$ block columns, i.e., the first nine $(n = 9)$ columns of the Kalman controllability matrix, we have an inverse matrix

$$\mathbf{\Omega}^{-1} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{1}{b_{41}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{b_{52}} & 0 & -\frac{a_{54}}{b_{52}} & 0 & 0 \\ 0 & \frac{1}{b_{23}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b_{41}} & 0 & 0 \\ -\frac{a_{54}}{a_{17}b_{52}} & 0 & 0 & 0 & 0 & \frac{a_{12}a_{54}}{a_{17}a_{62}b_{52}} & 0 & \frac{1}{b_{52}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{62}b_{23}} & 0 & 0 & 0 \\ \frac{1}{a_{17}b_{41}} & 0 & 0 & 0 & 0 & -\frac{a_{12}}{a_{17}a_{62}b_{41}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{33}a_{35}b_{52}} & 0 & 0 & 0 & 0 & -\frac{1}{a_{33}b_{52}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{a_{62}b_{23}} \end{bmatrix}.$$

Thus, all the conditions of Theorem 2, are satisfied, and the controller matrix $\mathbf{K}$ can be calculated using the generalized Bass — Gura block-matrix formula (14).

We calculate from (10) the block-matrix vector of coefficients of the original block-matrix characteristic polynomial of the state matrix $\mathbf{A}$:

$$\mathbf{P}^{T} = \left[\mathbf{P}_0 \mid \mathbf{P}_1 \mid \mathbf{P}_2\right] = -\left(\mathbf{\Omega}^{-1}\mathbf{A}^3\mathbf{B}\right)^{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_{54}\frac{b_{41}}{b_{52}} & -a_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We write the matrix of the second similarity transformation $\mathbf{T}_2 = \tilde{\tilde{\mathbf{\Omega}}}^{-1}$ from (11)

$$
\mathbf{T}_2 = \left[\begin{array}{c:c:c} \mathbf{P}_1 & \mathbf{P}_2 & \mathbf{I}_3 \\ \mathbf{P}_2 & \mathbf{I}_3 & \mathbf{0}_{3\times3} \\ \mathbf{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{array}\right] = \left[\begin{array}{c:c:c:c:c:c:c:c:c}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -a_{54}\dfrac{b_{41}}{b_{52}} & -a_{33} & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-a_{54}\dfrac{b_{41}}{b_{52}} & -a_{33} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

and its inverse matrix $\tilde{\tilde{\boldsymbol{\Omega}}} = \mathbf{T}_2^{-1}$ from (22)

$$
\mathbf{T}_2^{-1} = \left[\begin{array}{c:c:c} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_3 \\ \mathbf{0}_{3\times3} & \mathbf{I}_3 & -\mathbf{P}_2 \\ \mathbf{I}_3 & -\mathbf{P}_2 & \mathbf{P}_2^2 - \mathbf{P}_1 \end{array}\right] = \left[\begin{array}{c:c:c:c:c:c:c:c:c}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & a_{54}\dfrac{b_{41}}{b_{52}} & a_{33} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & a_{54}\dfrac{b_{41}}{b_{52}} & a_{33} & 0 & a_{33}a_{54}\dfrac{b_{41}}{b_{52}} & a_{33}^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

We determine the desired controller matrix $\mathbf{K}$ by the formula (14)

$$
\mathbf{K} = \left(\mathbf{P}^{*T} - \mathbf{P}^{T}\right)\mathbf{T}_2^{-1}\boldsymbol{\Omega}^{-1} =
$$

$$
= \left[\begin{array}{c:c:c:c:c:c:c:c:c}
\dfrac{f_3}{a_{17}b_{41}} & 0 & 0 & \dfrac{f_1}{b_{41}} & 0 & \dfrac{\tilde{c}}{a_{62}} & \dfrac{f_2}{b_{41}} & 0 & 0 \\
0 & \dfrac{\kappa m_1}{b_{23}} & \dfrac{s_{a_{33}}}{a_{33}a_{35}b_{52}} & \dfrac{a_{54}}{b_{52}} & \dfrac{a_{33}+s_1}{b_{52}} & \dfrac{\kappa m_2}{a_{62}b_{23}} & 0 & -\dfrac{s_3}{a_{33}b_{52}} & \dfrac{\kappa m_3}{a_{62}b_{23}} \\
0 & \dfrac{s_1}{b_{23}} & -\dfrac{m_{a_{33}}}{\kappa a_{33}a_{35}b_{52}} & 0 & -\dfrac{m_1}{\kappa b_{52}} & \dfrac{s_2}{a_{62}b_{23}} & 0 & \dfrac{m_3}{\kappa a_{33}b_{52}} & \dfrac{s_3}{a_{62}b_{23}}
\end{array}\right],
$$

where

$$s_{a_{33}} = a_{33}^3 + s_1 a_{33}^2 + s_2 a_{33} + s_3 = \frac{g_{a_{33}} + h_{a_{33}}}{2} \in \mathbb{R},$$

$$m_{a_{33}} = m_1 a_{33}^2 + m_2 a_{33} + m_3 = \frac{g_{a_{33}} - h_{a_{33}}}{2} i \in \mathbb{R},$$

$$g_{a_{33}} = a_{33}^3 + g_1 a_{33}^2 + g_2 a_{33} + g_3 = (a_{33} - \phi_4)(a_{33} - \phi_6)(a_{33} - \phi_8) = s_{a_{33}} - m_{a_{33}} i,$$

$$h_{a_{33}} = a_{33}^3 + h_1 a_{33}^2 + h_2 a_{33} + h_3 = (a_{33} - \phi_5)(a_{33} - \phi_7)(a_{33} - \phi_9) = s_{a_{33}} + m_{a_{33}} i,$$

$$\tilde{c} = \frac{c}{b_{23}} - \frac{f_3 a_{12}}{a_{17} b_{41}}.$$

Analyzing the form of the controller matrix that provides the specified pole placement, let us solve the additional problem of reducing its norm (the sum of squares of its coefficients). To do this, we find the values of the parameters c and κ, that afford a minimum to the norm of the matrix $\mathbf{K}$ as a function of the two indicated arguments. We consider only those coefficients of the matrix $\mathbf{K}$, that depend on the parameters c and κ.

The parameter c only affects the matrix $\mathbf{K}$ coefficient located in the sixth column of the first row. By assigning

$$c = \frac{a_{12}}{a_{17}} \frac{b_{23}}{b_{41}} f_3,$$

one can reset this ratio.

The parameter κ is located in the matrix $\mathbf{K}$ in the numerators of the coefficients of the second row (columns 2, 6 and 9), as well as in the denominators of the coefficients of the third row (columns 3, 5 and 8). The sum of squares of these coefficients is

$$S(\kappa) = \frac{\kappa^2}{b_{23}^2} \left(m_1^2 + \frac{m_2^2 + m_3^2}{a_{62}^2} \right) + \frac{1}{\kappa^2 b_{52}^2} \left(m_1^2 + \frac{m_{a_{33}}^2 + m_3^2 a_{35}^2}{a_{33}^2 a_{35}^2} \right).$$

The minimum of the function $S(\kappa)$ is achieved, when

$$\frac{dS}{d\kappa} = \frac{2\kappa}{b_{23}^2} \left(m_1^2 + \frac{m_2^2 + m_3^2}{a_{62}^2} \right) - \frac{2}{\kappa^3 b_{52}^2} \left(m_1^2 + \frac{m_{a_{33}}^2 + m_3^2 a_{35}^2}{a_{33}^2 a_{35}^2} \right) = 0,$$

i.e., when

$$\kappa = \pm \sqrt{\left| \frac{b_{23}}{b_{52}} \frac{a_{62}}{a_{33} a_{35}} \right| \sqrt{\frac{m_{a_{33}}^2 + \left(m_1^2 a_{33}^2 + m_3^2 \right) a_{35}^2}{m_1^2 a_{62}^2 + m_2^2 + m_3^2}}},$$

by virtue of the fact that

$$\frac{d^2S}{d\kappa^2}=\frac{2}{b_{23}^2}\left(m_1^2+\frac{m_2^2+m_3^2}{a_{62}^2}\right)+\frac{6}{\kappa^4 b_{52}^2}\left(m_1^2+\frac{m_{a_{33}}^2+m_3^2a_{35}^2}{a_{33}^2a_{35}^2}\right)>0.$$

We perform a check. We form the characteristic matrix of the closed-loop system object–controller

$$\mathbf{Z}=\lambda\mathbf{I}_9-\left(\mathbf{A}-\mathbf{BK}\right)=$$

$$=\left[\begin{array}{c:c:c:c:c:c:c:c:c}
\lambda & -a_{12} & 0 & 0 & 0 & 0 & -a_{17} & 0 & 0\\
0 & \lambda+s_1 & -\dfrac{m_{a_{33}}b_{23}}{\kappa a_{33}a_{35}b_{52}} & 0 & -\dfrac{m_1b_{23}}{\kappa b_{52}} & \dfrac{s_2}{a_{62}} & 0 & \dfrac{m_3b_{23}}{\kappa a_{33}b_{52}} & \dfrac{s_3}{a_{62}}\\
0 & 0 & \lambda-a_{33} & 0 & -a_{35} & 0 & 0 & 0 & 0\\
\dfrac{f_3}{a_{17}} & 0 & 0 & \lambda+f_1 & 0 & \dfrac{b_{41}\tilde{c}}{a_{62}} & f_2 & 0 & 0\\
0 & \dfrac{\kappa m_1b_{52}}{b_{23}} & \dfrac{s_{a_{33}}}{a_{33}a_{35}} & 0 & \lambda+a_{33}+s_1 & \dfrac{\kappa m_2b_{52}}{a_{62}b_{23}} & 0 & -\dfrac{s_3}{a_{33}} & \dfrac{\kappa m_3b_{52}}{a_{62}b_{23}}\\
0 & -a_{62} & 0 & 0 & 0 & \lambda & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & \lambda & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & \lambda & 0\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \lambda
\end{array}\right]$$

and we find its determinant. Having noted that for this matrix in block row 3

$$\mathbf{Z}_{[3,1]}=\mathbf{0}_{3\times3},\quad \mathbf{Z}_{[3,2]}=-\mathbf{I}_3,\quad \mathbf{Z}_{[3,3]}=\lambda\mathbf{I}_3,$$

and the block $\mathbf{Z}_{[1,1]}$ has a triangular form, we calculate $|\mathbf{Z}|$, successively applying the second and first Schur formulas on the determinants of block matrices [15]:

$$|\mathbf{Z}|=\left|\begin{bmatrix}\mathbf{Z}_{[1,1]} & \mathbf{Z}_{[1,2]}\\ \mathbf{Z}_{[2,1]} & \mathbf{Z}_{[2,2]}\end{bmatrix}-\begin{bmatrix}\mathbf{Z}_{[1,3]}\\ \mathbf{Z}_{[2,3]}\end{bmatrix}\mathbf{Z}_{[3,3]}^{-1}\left[\mathbf{Z}_{[3,1]} \;\vdots\; \mathbf{Z}_{[3,2]}\right]\right|\left|\mathbf{Z}_{[3,3]}\right|=$$

$$=\lambda^3\left|\begin{bmatrix}\mathbf{Z}_{[1,1]} & \mathbf{Z}_{[1,2]}+\lambda^{-1}\mathbf{Z}_{[1,3]}\\ \mathbf{Z}_{[2,1]} & \mathbf{Z}_{[2,2]}+\lambda^{-1}\mathbf{Z}_{[2,3]}\end{bmatrix}\right|=$$

$$=\left|\mathbf{Z}_{[1,1]}\right|\left|\lambda\left(\mathbf{Z}_{[2,2]}-\mathbf{Z}_{[2,1]}\mathbf{Z}_{[1,1]}^{-1}\mathbf{Z}_{[1,2]}\right)+\left(\mathbf{Z}_{[2,3]}-\mathbf{Z}_{[2,1]}\mathbf{Z}_{[1,1]}^{-1}\mathbf{Z}_{[1,3]}\right)\right|=$$

$$=\lambda\left(\lambda-a_{33}\right)\left(\lambda+s_1\right)\left|\begin{array}{c:c:c}
\dfrac{f_\lambda}{\lambda} & \dfrac{1}{b_{52}}\dfrac{1}{\lambda-a_{33}}\dfrac{a_{12}b_{23}}{a_{17}\kappa}\dfrac{m_\lambda}{\lambda+s_1}\dfrac{f_3}{\lambda} & \dfrac{1}{a_{62}}\left(\dfrac{b_{41}}{b_{23}}c\lambda-\dfrac{a_{12}}{a_{17}}\dfrac{s_\lambda}{\lambda+s_1}\dfrac{f_3}{\lambda}\right)\\
0 & \dfrac{1}{\lambda-a_{33}}\left(s_\lambda+\dfrac{m_1m_\lambda}{\lambda+s_1}\right) & \dfrac{b_{52}\kappa}{a_{62}b_{23}}\left(m_\lambda-\dfrac{m_1s_\lambda}{\lambda+s_1}\right)\\
0 & -\dfrac{a_{62}b_{23}}{b_{52}\kappa}\dfrac{1}{\lambda-a_{33}}\dfrac{m_\lambda}{\lambda+s_1} & \dfrac{s_\lambda}{\lambda+s_1}
\end{array}\right|=$$

$$=f_\lambda\left(s_\lambda^2+m_\lambda^2\right)=\prod_{\eta=1}^{9}\left(\lambda-\phi_\eta\right).$$

Thus,

$$\operatorname{eig}(\mathbf{A}-\mathbf{BK})=\{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\ \phi_5,\ \phi_6,\ \phi_7,\ \phi_8,\ \phi_9\}=\Lambda^*,$$

and the calculated controller by the state with the matrix $\mathbf{K}$ solves the posed problem of poles placement. Moreover, it is shown how, using the parameterization of the desirable block-matrix characteristic polynomial, it is possible to reduce the norm of the matrix $\mathbf{K}$ (the sum of squares of its coefficients) in comparison with the value of this norm without parameterization (for $c=0$ and $\kappa=1$).

Conclusion. The compact analytical formula for calculating the controller matrix is obtained for solving the synthesis problem of the modal controller providing desired pole placement by means of the fully measured state vector in linear dynamic systems with vector control. This formula represents a generalization of the known Bass — Gura formula used for similar systems with scalar control. In the proof of the corresponding theorem, non-singular similarity transformations and operations with block matrices were used, in particular, block transposition. However, the final formula does not require additional reduction of the original system to special canonical forms and can be directly applied to systems which have state-space dimension divisible by the number of control inputs and which have the first block columns of the Kalman controllability matrix (by a number corresponding to this multiplicity) forming a non-singular matrix. The generalized block-matrix Bass — Gura formula gives a countably-infinite set of solutions to the same problem of the desired pole placement under vector control and can be used to simultaneously solve other special problems, for example, to minimize the norm of the controller matrix. Numerical and analytical examples of the use of the generalized block-matrix Bass — Gura formula for the described class of systems with vector control are considered. They confirm the possibility of applying this formula regardless of the ratio of the dimensions of the state and control vectors, as well as on the presence and multiplicity of real poles or complex-conjugate pairs of poles in the original and desired spectra of the state matrix.

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