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СТАБИЛИЗАЦИЯ И СПЕКТР В ОПЕРАТОРНО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЯХ

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Изучена задача Коши для линейных операторно-дифференциальных уравнений второго порядка в гильбертовом пространстве. Рассмотрен случай неограниченного самосопряженного неотрицательного оператора, особое внимание уделено оператору Лапласа с краевым условием Дирихле. Исследована смешанная задача для волнового уравнения, введен энергетический класс решений и доказано представление решений в виде интеграла Бохнера – Стилтьеса. Установлена связь между спектральными свойствами оператора Лапласа и стабилизацией при больших значениях времени решений смешанной задачи для волнового уравнения. Исследовано асимптотическое поведение по времени функции локальной энергии для различных типов спектра. В случае ограниченных областей, когда оператор Лапласа имеет дискретный спектр, доказано, что решение, локальная энергия которого стремится к нулю, равно нулю тождественно. Для произвольных областей в случае оператора с непустым точечным спектром доказано существование гладких и финитных начальных функций, для которых функция локальной энергии не стремится к нулю. Показано, что для оператора с непрерывным спектром стремится к нулю среднее по времени функции локальной энергии. Для случая абсолютно непрерывного спектра установлено стремление к нулю самой функции локальной энергии.

Ключевые слова: операторно-дифференциальное уравнение, гиперболическое уравнение, краевое условие Дирихле, стабилизация, оператор Лапласа, спектр.

STABILIZATION AND SPECTRUM IN OPERATOR DIFFERENTIAL EQUATIONS

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We investigate the Cauchy problem for a second order non-stationary linear operator differential equation in a Hilbert space. We consider the case of an unbounded self-adjoint positive operator with a special regard to the Laplace operator with Dirichlet boundary conditions. The corresponding problem is a mixed problem for a wave equation. Introducing the energy class solution we prove its representation by the Bochner – Stiltjes integral. We establish the connection between spectral properties of the Laplace operator and stabilization for large time values of the solutions to the mixed problem of the wave equation. We investigate the asymptotic behavior in time of the local energy function for the various types of spectrum. For bounded domains where the spectrum of the Laplace operator is purely discrete we any solution with *a local energy tends to zero in time is identically zero. For arbitrary domains in the case of operator with non-empty point spectrum we prove that there are smooth and finite initial functions for which the local energy function does not decay. In the cases of continuous and absolutely continuous spectrum of the Laplace operator we prove the mean decay and the decay of the local energy function respectively.*

Keywords**:** operator differential equation, hyperbolic problem, Dirichlet boundary condition, stabilization, Laplace operator, spectrum.

Introduction. Very often in mathematical physics arises the question of large time behavior for the solutions of Cauchy problem for the non-stationary operator equation

$$u_{tt} + Lu = 0, \quad t > 0; \tag{1}$$

$$u|_{t=0} = f, \quad u_t\Big|_{t=0} = g, \tag{2}$$

where L is a linear self-adjoint operator in a Hilbert space H. The interest of mathematicians to this problem is natural because many important physical problems leads to the Cauchy problem (1), (2). The examples of such problems are acoustic and electromagnetic oscillations in homogeneous and non-homogeneous media [1, 2]. Closely related operator equations arises in the problem of small vibrations of an ideal non-homogeneous fluid [3].

The general theory of the operator Cauchy problem (1), (2) in a Hilbert space contains many results about solvability and a priory estimates for solutions [4–9]. We will investigate qualitative properties of solutions of the problem (1), (2) with special attention to the connection between spectral properties of operator L and behavior of solutions for $t \to \infty$.

Let us note that the existence results and qualitative properties of solutions to the problem (1), (2) closely connected with the representation formulas for solutions. So, one of the first results concerned to the case of a bounded positive operator L [10] state that the solution of (1), (2) represents by the series $u(t) = \cos(\sqrt{L}t)f + \dfrac{\sin(\sqrt{L}t)}{\sqrt{L}}g$, where

$$\cos(\sqrt{L}t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k} L^k}{(2k)!!}; \quad \sin(\sqrt{L}t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1} L^{k+\frac{1}{2}}}{(2k+1)!!}.$$

But more interesting due to physical and technical applications is the case of unbounded operator L. Let L is an unbounded positive operator whose domain $D(L)$ is dense in H. We suppose that the inverse operator L^{-1} is bounded. We formulate the solvability result for the non-homogeneous problem

$$Su = u_{tt} + Lu = h, \quad t > 0; \tag{3}$$

$$u|_{t=0} = f, \quad u_t|_{t=0} = g. \tag{4}$$

Let H_1 be a Hilbert space of measurable functions $u(t) : [0, T] \to H$ with a scalar product $(u, v)_{H_1} = \int\limits_0^T (u(t), v(t))_H \, dt$ and the norm $\|u\|^2_{H_1} = (u, u)_{H_1}$. We say that the function $u(t)$ has a derivative on $[0, T]$ if the following representation holds:

$$u(t) = u(t_0) + \int\limits_{t_0}^{t} v(\tau) dt, \quad t_0 \in [0, T]. \tag{5}$$

Here the function $v(t) \in H_1$ gives for almost all $t \in [0, T]$ the value of the derivative du/dt. Consider an operator $Au = (Su, u(0), u_t(0))$. The operator A defines on the domain $D(A)$ of all $u \in H_1$ such that functions du/dt, Lu, Ldu/dt belong to H_1 and continuous; the functions $\frac{d^2u}{dt^2}$, $L\frac{d^2u}{dt^2}$ belong to H_1 and piecewise continuous. The image of operator $R(A)$ is a linear manifold in a Hilbert space $W = H_1 \times D(\sqrt{L}) \times H$ where $D(\sqrt{L})$ is a Hilbert space of elements $u = L^{-\frac{1}{2}}\psi$, $\psi \in H$ with the scalar product $(u, v)_{D(\sqrt{L})} = \left(\sqrt{L}u, \sqrt{L}v\right)$. It is possible to consider the solution of the problem (3), (4) as a solution of the operator equation $Au = (h, f, g)$, where $(h, f, g) \in W$.

Theorem 1. *The operator A admits a closure $\overline{A}$, $R(\overline{A}) = \overline{R(A)} = W$. There exists a bounded inverse operator $\overline{A}^{-1}$ on W and the problem*

$$\overline{A}u = (h, f, g) \tag{6}$$

has the unique solution for all $h \in H_1$, $f \in D(\sqrt{L})$, $g \in H$ [8].

It is possible to prove that the solution of the operator equation (6) is a weak solution of the problem (3), (4). It means that $u \in H_1$, $u_t \in H_1$, $\sqrt{L}u \in H_1$, $u(0) = f$ and the following integral identity holds:

$$\int\limits_0^T \left((u_t, w_t) - \left(\sqrt{L}u, \sqrt{L}w\right)\right) dt + (g, w(0)) = 0 \tag{7}$$

for all $w \in H_1$, $w_t \in H_1$, $\sqrt{L}w \in H_1$, $w(T) = 0$. The solution from such class is unique.

Below we consider the case of hyperbolic problems (1), (2). The main example of an elliptic second-order operator L is the Laplace operator $Lu = -\Delta u$. Let $\Omega \subset R^n$, $n \geq 2$, be an an arbitrary (may be unbounded) domain with smooth boundary Γ. We consider the mixed problem for the wave equation

$$u_{tt} - \Delta u = 0 \tag{8}$$

in $(t > 0) \times \Omega$, with the initial conditions

$$u(0,x) = f(x); \quad u_t(0,x) = g(x) \quad \text{for} \quad x \in \Omega \tag{9}$$

and the boundary condition

$$u = 0 \tag{10}$$

on $(t > 0) \times \Gamma$. We assume the initial functions $f \in \overset{o}{H}{}^1(\Omega)$ and $g \in L_2(\Omega)$ are real-valued.

It is well known that a solution of the problem (8)–(10) satisfies the energy conservation law

$$\mathcal{E}(t) = \|u_t\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} = \mathcal{E}(0) \tag{11}$$

for $t > 0$.

Below we study the connections between spectral properties of the Laplace operator and the behavior for large values of time to solutions of the problem (8)–(10). We investigate stabilization in time with special regard to the properties of the local energy function

$$\mathcal{E}_{\Omega'}(t) = \|u_t\|^2_{L_2(\Omega')} + \|\nabla u\|^2_{L_2(\Omega')} \tag{12}$$

where $\Omega' \subset \Omega$ is a bounded domain.

Let us note that many practically important problems deal with unbounded domains Ω so the inverse operator L^{-1} may be unbounded in the main space $H = L_2(\Omega)$.

Solutions from the Energy Class and Spectral Representation. We consider solutions of the problem (8)–(10) from the energy class (see [11]), that is a function $u(t,x) \in C([0,+\infty); \overset{o}{H}{}^1(\Omega))$ such that $u_t(t,x) \in$ $\in C([0,+\infty); L_2(\Omega))$ satisfying the initial conditions (9), the equality (11) and the integral identity

$$\int_0^T \int_\Omega ((\nabla u, \nabla w) - u_t w_t) dt\, dx = \int_\Omega g(x) w(0,x)\, dx,$$

for all $w \in H^1((0,T) \times \Omega)$ satisfying $w = 0$ on $(0,T) \times \Gamma$ and $w(T,x) = 0$, $T > 0$.

Consider a self-adjoint non-negative operator $L : D(L) \to L_2(\Omega)$ generated in $L_2(\Omega)$ space by differential expression $-\Delta$ with Dirichlet boundary condition. Using an estimates to solutions of elliptic boundary value problems [14], we have the domain of operator L: $D(L) = \overset{o}{H}{}^1(\Omega) \cap$ $\cap H^2(\Omega') \cap \{\Delta u \in L_2(\Omega)\}$, here $\Omega' \subset \Omega$ is an arbitrary bounded domain. Let $\{E(\lambda)\}$, $-\infty < \lambda < +\infty$, be a family of spectral projectors associated with operator L [15].

Lemma 1. *Let* $f \in \overset{o}{H}{}^1(\Omega)$, $g \in L_2(\Omega)$. *Then the solution of the problem* (8)–(10) *from the energy class can be written as Bochner–Stiltjes integral*

$$u = \int_0^\infty \cos(\sqrt{\lambda}t)dE(\lambda)f + \int_0^\infty \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dE(\lambda)g. \quad (13)$$

The integral (13) converges uniformly in $\overset{o}{H}{}^1(\Omega)$ *on* $[0,T]$, $T > 0$. *The derivative* u_t *can be written as*

$$u_t = -\int_0^\infty \sqrt{\lambda}\sin(\sqrt{\lambda}t)\,dE(\lambda)f + \int_0^\infty \cos(\sqrt{\lambda}t)\,dE(\lambda)g, \quad (14)$$

the integral (14) *converges uniformly in* $L_2(\Omega)$ *on* $[0,T]$.

Point Spectrum and Non-Decay of Local Energy. In a bounded domain Ω the spectrum of operator L is discrete. So, the solution of the problem (8)–(10) represents by the series

$$u(t,x) = \sum_{j=1}^\infty (a_j \cos(\sqrt{\lambda_j}t) + b_j \frac{\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}})v_j(x), \quad (15)$$

where $a_j = (f, v_j)_{L_2(\Omega)}$, $b_j = (g, v_j)_{L_2(\Omega)}$. Here $0 < \lambda_1 < \lambda_2 \le \ldots$, $\lim\limits_{j\to\infty} \lambda_j = +\infty$, is the sequence of eigenvalues of the operator L, $\{v_j\}$ is the orthonormal basis of eigenfunctions in $L_2(\Omega)$.

Using the equality (15), we can prove that solution of the problem (8)–(10) is an almost-periodic function with respect to t ($n = 1$ [16]; $n \ge 2$ [18, 19]).

Theorem 2. *Let* $\Omega \subset R^n$ *be a bounded domain and* $\lim\limits_{t\to\infty} \mathcal{E}_{\Omega'}(t) = 0$ *for some domain* $\Omega' \subset \Omega$. *Then* $u = 0$ *in* $(0,\infty) \times \Omega$.

◀ The equality $\lim\limits_{t\to\infty} \mathcal{E}_{\Omega'}(t) = 0$ means that $\lim\limits_{t\to\infty} \|\nabla u(t,x)\|_{L_2(\Omega')} = 0$. Therefore,

$$0 = \lim_{t\to\infty} \int_{\Omega'} u_{x_k}(t,x)\eta(x)dx =$$

$$= \lim_{t\to\infty} \sum_{j=1}^\infty \Big(a_j \cos(\sqrt{\lambda_j}t) + b_j \frac{\sin(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}}\Big) \int_{\Omega'} (v_j)_{x_k}(x)\eta(x)dx, \ k = 1, \ldots, n,$$

for an arbitrary function $\eta(x) \in \mathcal{D}(\Omega') = \overset{o}{C}{}^\infty(\Omega')$. We have

$$0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\Omega'} u_{x_k}(t,x)\eta(x)dx \cos(\sqrt{\lambda_m}t)dt =$$

$$= \frac{1}{2} \int_{\Omega'} \sum_{\lambda_j=\lambda_m} a_j (v_j)_{x_k}(x)\eta(x)\,dx; \quad (16)$$

$$0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\Omega'} u_{x_k}(t,x)\eta(x)dx \sin(\sqrt{\lambda_m}t)dt =$$

$$= \frac{1}{2} \int_{\Omega'} \sum_{\lambda_j=\lambda_m} \frac{b_j}{\sqrt{\lambda_j}} (v_j)_{x_k}(x)\eta(x)\, dx \quad (17)$$

for $m = 1, 2, \ldots$ The equalities (16), (17) mean that

$$\sum_{\lambda_j=\lambda_m} a_j(v_j)_{x_k}(x) = 0; \qquad \sum_{\lambda_j=\lambda_m} \frac{b_j}{\sqrt{\lambda_j}}(v_j)_{x_k}(x) = 0 \quad (18)$$

for $x \in \Omega'$ and $m = 1, 2, \ldots$ Using the analyticity of eigenfunctions in Ω we obtain from (18) that

$$\sum_{\lambda_j=\lambda_m} a_j(v_j)_{x_k}(x) = 0; \qquad \sum_{\lambda_j=\lambda_m} \frac{b_j}{\sqrt{\lambda_j}}(v_j)_{x_k}(x) = 0, \quad k = 1, \ldots, n,$$

for $x \in \Omega$ and $m = 1, 2, \ldots$ It now follows from the boundary condition (10) that $\sum_{\lambda_j=\lambda_m} a_j v_j(x) = 0$, $\sum_{\lambda_j=\lambda_m} \frac{b_j}{\sqrt{\lambda_j}} v_j(x) = 0$ for $x \in \Omega$ and $m = 1, 2, \ldots$ By the orthogonality of eigenfunctions we have $a_j = b_j = 0$ for all $j : \lambda_j = \lambda_m$, $m = 1, 2, \ldots$ Proof of Theorem 2 is complete. ►

In the case of unbounded domain we say that the energy scatters to infinity if for any bounded $\Omega' \subset \Omega$

$$\lim_{t\to\infty} \mathcal{E}_{\Omega'}(t) = 0. \quad (19)$$

The following theorem means that in the case of $\sigma_p(L) \neq \varnothing$ (the continuous spectrum $\sigma_c(L)$ can be non-empty too) the relation (19) does not holds even for smooth and finite initial functions.

Theorem 3. *Let $\sigma_p(L) \neq \varnothing$. Then there exist the functions $f, g \in \mathcal{D}(\Omega)$ and domain Ω' is compact embedded to Ω such that*

$$\liminf_{t\to\infty} \mathcal{E}_{\Omega'}(t) > 0. \quad (20)$$

◄ Let $\lambda \in \sigma_p(L)$, $v(x) \in \overset{o}{H}{}^1(\Omega)$ is a corresponding eigenfunction. It is sufficient to consider $\lambda > 0$ because $0 \notin \sigma_p(L)$. Really, for $\lambda = 0$ an eigenfunction is a harmonic function from $\overset{o}{H}{}^1(\Omega)$ and vanishes in Ω [2, Ch. 2, Par. 4, no. 4.7]. Consider the solution $u(t,x) = \cos(\sqrt{\lambda}t)v(x)$ of the problem (8)–(10) with $f = v$, $g = 0$ we obtain

$$\mathcal{E}_{\Omega'}(t) = \frac{1}{2} \int_{\Omega'} \left(|\nabla v|^2 + \lambda v^2\right) dx + \frac{\cos(2\sqrt{\lambda}t)}{2} \int_{\Omega'} \left(|\nabla v|^2 - \lambda v^2\right) dx. \quad (21)$$

Now, we have an integral equality for an eigenfunction: $\int\limits_{\Omega} |\nabla v|^2\, dx = \lambda \int\limits_{\Omega} v^2\, dx$. Therefore,

$$\int\limits_{\Omega} \left(|\nabla v|^2 + \lambda v^2\right) dx > 0; \quad \int\limits_{\Omega} \left(|\nabla v|^2 - \lambda v^2\right) dx = 0$$

and by the absolute continuity of the Lebesgue integral there exists a domain Ω' is compact embedded to Ω such that for some $\varepsilon > 0$ we have the inequalities

$$\int\limits_{\Omega'} \left(|\nabla v|^2 + \lambda v^2\right) dx \geq \varepsilon; \quad \left|\int\limits_{\Omega'} \left(|\nabla v|^2 - \lambda v^2\right) dx\right| \leq \frac{\varepsilon}{2}. \tag{22}$$

Thus, by (21), (22) we obtain an inequality

$$\mathcal{E}_{\Omega'}(t) \geq \frac{\varepsilon}{4}$$

for $t > 0$. Let us consider the solution $\tilde{u}$ of the problem (8)–(10) with initial functions $f = \tilde{v}$, $g = 0$, where function $\tilde{v} \in \mathcal{D}(\Omega)$ satisfy the inequality $\|v - \tilde{v}\|^2_{H^1(\Omega)} < \varepsilon/16$. Then

$$\int\limits_{\Omega'} (\tilde{u}_t^2 + |\nabla \tilde{u}|^2)\, dx \geq \frac{1}{2} \int\limits_{\Omega'} \left(u_t^2 + |\nabla u|^2\right) dx -$$
$$- \int\limits_{\Omega'} (((u - \tilde{u})_t)^2 + |\nabla (u - \tilde{u})|^2)\, dx \geq$$
$$\geq \frac{\varepsilon}{8} - \int\limits_{\Omega'} (((u - \tilde{u})_t)^2 + |\nabla (u - \tilde{u})|^2)\, dx \geq$$
$$\geq \frac{\varepsilon}{8} - \int\limits_{\Omega} (((u - \tilde{u})_t)^2 + |\nabla (u - \tilde{u})|^2)\, dx =$$
$$= \frac{\varepsilon}{8} - \|\nabla (v - \tilde{v})\|^2_{L_2(\Omega)} > \frac{\varepsilon}{8} - \frac{\varepsilon}{16} = \frac{\varepsilon}{16} > 0, \quad t > 0.$$

The inequality (20) is proved. ►

Continuity of Spectrum and Mean Decay of Local Energy. Let Ω be an unbounded domain. In the case of $\sigma_p(L) = \varnothing$ we have the mean local energy decay. A proof use the following theorem [1, Th. 9.15, 20].

Theorem 4. *Let real-valued function* $m(z) \in C(-\infty, +\infty)$, $m(z) = 0$ *for* $z \leq 0$, $\mathrm{var}_{[0,+\infty)} m(z) < \infty$ *and* $s(t) = \int\limits_0^{\infty} e^{izt}\, dm(z)$ *for* $t > 0$. *Then*

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t |s(\tau)|^2\,d\tau = 0. \tag{23}$$

Now we prove the main result of this section.

Theorem 5. *Let $\sigma_p(L)=\varnothing$. Then for all $f\in \overset{o}{H}{}^1(\Omega)$, $g\in L_2(\Omega)$ and all bounded domains $\Omega'\subset\Omega$*

$$\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t \mathcal{E}_{\Omega'}(\tau)\,d\tau = 0. \tag{24}$$

◀ It is sufficient to prove the relation (24) for $f,g\in\mathcal{D}(\Omega)$. For any $f\in \overset{o}{H}{}^1(\Omega)$, $g\in L_2(\Omega)$ and arbitrary $\varepsilon>0$ there exist $\tilde f,\tilde g\in\mathcal{D}(\Omega)$ such that $\|f-\tilde f\|_{H^1(\Omega)}<\varepsilon$, $\|g-\tilde g\|_{L_2(\Omega)}<\varepsilon$. Therefore, for $\Omega'=\Omega_R=\Omega\cap\{|x|< <R\}$ we have

$$\mathcal{E}_{\Omega_R}(t)=\int\limits_{\Omega_R}\left(u_t^2+|\nabla u|^2\right)dx\le 2\int\limits_{\Omega_R}\left(\tilde u_t^2+|\nabla\tilde u|^2\right)dx+$$
$$+2\int\limits_{\Omega}\left(\left((u-\tilde u)_t\right)^2+|\nabla(u-\tilde u)|^2\right)dx = 2\int\limits_{\Omega_R}\left(\tilde u_t^2+|\nabla\tilde u|^2\right)dx+$$
$$+2\left(\|f-\tilde f\|^2_{H^1(\Omega)}+\|g-\tilde g\|^2_{L_2(\Omega)}\right)<2\int\limits_{\Omega_R}\left(\tilde u_t^2+|\nabla\tilde u|^2\right)dx+4\varepsilon^2$$

for $t>0$. Now, to prove (24) suppose that $f,g\in\mathcal{D}(\Omega)$. Note that $f,g\in\mathcal{D}(\Omega)\subset D(L^p)$, $p=1,2,\ldots$, where $D(L^p)$ is a domain of the p-th power of the Laplace operator $L=-\Delta$ with Dirichlet boundary condition. Thence [17, Ch. 9, Par. 1] for $f,g\in\mathcal{D}(\Omega)$ we have the inequalities

$$\int\limits_0^\infty \lambda^{2p}\,d(E(\lambda)f,f)<\infty;\quad \int\limits_0^\infty \lambda^{2p}\,d(E(\lambda)g,g)<\infty,\quad p=0,1,2,\ldots \tag{25}$$

Now, by the operator calculus for self-adjoint operators [17, Ch. 9, Par. 1] for $q(x)\in\mathcal{D}(\Omega)$ and $t>0$ we obtain the relations

$$(u_t,q)_{L_2(\Omega)}=-\int\limits_0^\infty \sin(\sqrt{\lambda}t)\sqrt{\lambda}\,d(E(\lambda)f,q)+\int\limits_0^\infty\cos(\sqrt{\lambda}t)\,d(E(\lambda)g,q)=$$
$$=\int\limits_0^\infty \sin zt\,dm_1(z)+\int\limits_0^\infty\cos zt\,dm_2(z); \tag{26}$$

$$(\nabla u, \nabla q)_{L_2(\Omega)} = \left(\sqrt{L}u, \sqrt{L}q\right)_{L_2(\Omega)} =$$

$$= \int_0^\infty \lambda \cos(\sqrt{\lambda}t)\, d(E(\lambda)f, q) + \int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda}t)\, d(E(\lambda)g, q) =$$

$$= \int_0^\infty \cos zt\, dm_3(z) + \int_0^\infty \sin zt\, dm_4(z). \quad (27)$$

Here we have the equalities

$$dm_1(z) = -zd(E(z^2)f, q); \quad dm_2(z) = d(E(z^2)g, q);$$
$$dm_3(z) = z^2 d(E(z^2)f, q); \qquad dm_4(z) = zd(E(z^2)g, q).$$

The operator L is a positive operator with $\sigma_p(L) = \varnothing$, so $m_j(z)$ are continuous functions for $-\infty < z < +\infty$ and $m_j(z) = 0$ for $z < 0$. By the inequalities (25) and the relation [17, Ch. 9, Par. 1, Pt. 128, Eq. (12)] we obtain that

$$\mathrm{var}_{[0,+\infty)} m_j(z) < \infty, \quad j = 1, \ldots, 4. \quad (28)$$

It follows from (23)–(26) that for all $q(x) \in \mathcal{D}(\Omega)$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (u_\tau(\tau, x), q(x))^2_{L_2(\Omega)}\, d\tau = 0; \quad (29)$$

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (\nabla u(\tau, x), \nabla q(x))^2_{L_2(\Omega)}\, d\tau = 0. \quad (30)$$

Consider the closure of this equalities on q in $L_2(\Omega)$, we obtain that (29) is valid for all functions $q \in L_2(\Omega)$ and (30) — for all $q \in L_2(\Omega) \cap H^1(\Omega_R)$ for any $R > 0$, satisfying the condition $q = 0$ on Γ and such that $\int_\Omega |\nabla q|^2\, dx < \infty$. Furthermore, we have

$$\|\Delta u\|^2_{L_2(\Omega)} + \|\nabla u_t\|^2_{L_2(\Omega)} =$$

$$= \int_0^\infty \lambda^2 \cos^2(\sqrt{\lambda}t)\, d(E(\lambda)f, f) + \int_0^\infty \lambda \sin^2(\sqrt{\lambda}t)\, d(E(\lambda)g, g) +$$

$$+ \int_0^\infty \lambda^2 \sin^2 \sqrt{\lambda}t\, d(E(\lambda)f, f) + \int_0^\infty \lambda \cos^2 \sqrt{\lambda}t\, d(E(\lambda)g, g) =$$

$$= \int_0^\infty \lambda^2 d(E(\lambda)f, f) + \int_0^\infty \lambda d(E(\lambda)g, g) =$$

$$= \|\Delta f\|^2_{L_2(\Omega)} + \|\nabla g\|^2_{L_2(\Omega)}. \quad (31)$$

Applying Friedrichs inequality with an arbitrary $R > 0$ we obtain an estimate $\|u\|_{L_2(\Omega_R)} \le C(R)\|\nabla u\|_{L_2(\Omega_R)}$ for $t > 0$. Thence, holds the following inequality:

$$\|\nabla u\|_{H^1(\Omega_R)} + \|u_t\|_{H^1(\Omega_R)} \le C(R). \tag{32}$$

So, the set of functions $\{u_t(t,x)\}$ and $\{\nabla u(t,x)\}$, $t > 0$, are compact in $L_2(\Omega_R)$ for any $R > 0$. Let us prove that $\lim\limits_{t\to\infty} \dfrac{1}{t}\int\limits_0^t \|u_\tau\|^2_{L_2(\Omega_R)}\,d\tau = 0$.

By the estimate (32) the set of functions $\{u_t(t,x)\}$, $t > 0$ is compact in $L_2(\Omega_R)$. Let $\{h_{j,R}(x)\}$, $j = 1, 2, \ldots$, $x \in \Omega_R$ be an orthonormal basis in the $L_2(\Omega_R)$ space. We continue the functions $h_{j,R}$ by zero to $\Omega\backslash\Omega_R$. Denote the continued functions as $h_{j,R}$ too. Then $u_t(t,x) = \sum\limits_{j=1}^{\infty} c_{j,R}(t)h_{j,R}(x)$ for $t > 0$. We have $\lim\limits_{N\to\infty} \|u_t - \sum\limits_{j=1}^{N} c_{j,R}(t)h_{j,R}(x)\|_{L_2(\Omega_R)} = 0$ for $t > 0$. By the compactness criterion [21, P. 247, Th. 3] in the space $L_2(\Omega_R)$ with basis $\{h_{j,R}\}$ for all $\varepsilon > 0$ there exists $N > 0$ such that $u_t(t,x) = \sum\limits_{j=1}^{N} c_{j,R}(t)h_{j,R}(x) + \sum\limits_{j=N+1}^{\infty} c_{j,R}(t)h_{j,R}(x)$ for $t > 0$, $x \in \Omega_R$ and $\|\sum\limits_{j=N+1}^{\infty} c_{j,R}h_{j,R}\|_{L_2(\Omega_R)} < \varepsilon$, $t > 0$. Thence,

$$\|u_t\|^2_{L_2(\Omega_R)} = \|\sum_{j=1}^{N} c_{j,R}h_{j,R}\|^2_{L_2(\Omega_R)} + \|\sum_{j=N+1}^{\infty} c_{j,R}h_{j,R}\|^2_{L_2(\Omega_R)} =$$
$$= \sum_{j=1}^{N} c^2_{j,R}(t) + \|\sum_{j=N+1}^{\infty} c_{j,R}h_{j,R}\|^2_{L_2(\Omega)} < \sum_{j=1}^{N} c^2_{j,R}(t) + \varepsilon^2. \tag{33}$$

Integrate (33), we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int\limits_0^t \|u_\tau\|^2_{L_2(\Omega_R)}\,d\tau < \sum_{j=1}^{N}\left(\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t c^2_{j,R}\,d\tau\right) + \varepsilon^2 =$$
$$= \sum_{j=1}^{N}\left(\lim_{t\to\infty}\frac{1}{t}\int\limits_0^t (u_\tau, h_{j,R})^2_{L_2(\Omega_R)}\,d\tau\right) + \varepsilon^2. \tag{34}$$

By the equality (29) we have $\lim\limits_{t\to\infty}\dfrac{1}{t}\int\limits_0^t (u_\tau, h_{j,R})^2_{L_2(\Omega)}\,d\tau = 0$ for $j =$

$= 1, 2, \ldots, N$ and it follows from (34) that for any $\varepsilon > 0$ $\limsup\limits_{t\to\infty} \frac{1}{t}\int\limits_0^t \|u_\tau\|^2_{L_2(\Omega_R)}\, d\tau \le \varepsilon^2$. Now, we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int\limits_0^t \|u_\tau\|^2_{L_2(\Omega_R)}\, d\tau = 0 \tag{35}$$

for any $R > 0$. Let us prove now that $\lim\limits_{t\to\infty} \frac{1}{t}\int\limits_0^t \|\nabla u\|^2_{L_2(\Omega_R)}\, d\tau = 0$.

For any $R > 0$ we define the space: $\widetilde{H}_R = \big\{v \in H^1(\Omega_R) : v|_{\Gamma_R} = 0\big\}$, where $\Gamma_R = \Gamma \cap \{|x| < R\}$. The space $\widetilde{H}_R$ is a Hilbert space with a scalar product $(v, w)_{\widetilde{H}_R} = \int\limits_{\Omega_R} (\nabla v, \nabla w)\, dx$. Similarly we define the Hilbert space $\widetilde{H} = \{v \in H^1(\Omega_R)$ for any $R > 0 : v|_\Gamma = 0, \int\limits_\Omega |\nabla v|^2\, dx < \infty\}$, with a scalar product $(v, w)_{\widetilde{H}} = \int\limits_\Omega (\nabla v, \nabla w)\, dx$.

Let $R > 0$. It follows from (32) that the set of functions $\{u(t, x)\}$, $t > 0$, is compact in the space $\widetilde{H}_R$. Denote by $\{h_{j,R}(x)\}$, $j = 1, 2, \ldots$, the orthonormal basis in the space $\widetilde{H}_R$. By the compactness criterion in the space $\widetilde{H}_R$ with basis $\{h_{j,R}\}$ for any $\varepsilon > 0$ there exists $N > 0$ such that

$$u(t, x) = \sum_{j=1}^N b_{j,R}(t) h_{j,R}(x) + \sum_{j=N+1}^\infty b_{j,R}(t) h_{j,R}(x),$$

and $\|\sum\limits_{j=N+1}^\infty b_{j,R}(t) h_{j,R}(x)\|_{\widetilde{H}_R} < \varepsilon$ for $t > 0$. Now we have

$$\|u(t, x)\|^2_{\widetilde{H}_R} = \|\sum_{j=1}^N b_{j,R} h_{j,R}\|^2_{\widetilde{H}_R} + \|\sum_{j=N+1}^\infty b_{j,R} h_{j,R}\|^2_{\widetilde{H}_R} <$$

$$< \sum_{j=1}^N b^2_{j,R}(t) + \varepsilon^2 = \sum_{j=1}^N (u, h_{j,R})^2_{\widetilde{H}_R} + \varepsilon^2. \tag{36}$$

For all $v(x) \in \widetilde{H}$ we have an estimates $|F_j[v]| = \Big|(v, h_{j,R})_{\widetilde{H}_R}\Big| \le$ $\le \|v\|_{\widetilde{H}_R} \|h_{j,R}\|_{\widetilde{H}_R} = \|v\|_{\widetilde{H}_R} \le \|v\|_{\widetilde{H}}$, $j = 1, 2, \ldots, N$. It means that the linear functional $F_j[v] = (v, h_{j,R})_{\widetilde{H}_R}$ is a bounded functional on $\widetilde{H}$. By the F. Riesz theorem there exist functions $\tilde{h}_{j,R}(x) \in \widetilde{H}$, $j = 1, 2, \ldots, N$, such that

$$F_j[v] = \left(v, \tilde{h}_{j,R}\right)_{\widetilde{H}} = \left(\nabla v, \nabla \tilde{h}_{j,R}\right)_{L_2(\Omega)} = \int\limits_{\Omega} \left(\nabla v, \nabla \tilde{h}_{j,R}\right) dx. \quad (37)$$

It follows from (30) and (37) that

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t (u, h_{j,R})^2_{\widetilde{H}_R}\, d\tau = \lim_{t\to\infty} \frac{1}{t} \int\limits_0^t \left(u, \tilde{h}_{j,R}\right)^2_{\widetilde{H}} dt =$$

$$= \lim_{t\to\infty} \frac{1}{t} \int\limits_0^t \left(\nabla u, \nabla \tilde{h}_{j,R}\right)^2_{L_2(\Omega)} d\tau = 0 \quad (38)$$

for $j = 1, 2, \ldots, N$. Using (36) and (38), we obtain for any $\varepsilon > 0$ the relation

$$\limsup_{t\to\infty} \frac{1}{t} \int\limits_0^t \|\nabla u\|^2_{L_2(\Omega_R)}\, d\tau = \limsup_{t\to\infty} \frac{1}{t} \int\limits_0^t \|u\|^2_{\widetilde{H}_R}\, dt \leqslant$$

$$\leqslant \sum_{j=1}^{N} \left(\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t (u, \tilde{h}_{j,R})^2_{\widetilde{H}}\, dt \right) + \varepsilon^2 = \varepsilon^2.$$

So, we have the equality

$$\lim_{t\to\infty} \frac{1}{t} \int\limits_0^t \|\nabla u(\tau, x)\|^2_{L_2(\Omega_R)}\, d\tau = 0 \quad (39)$$

for any $R > 0$. Now, for any bounded $\Omega' \subset \Omega$ we take R sufficiently large such that $\Omega' \subset \Omega_R$. Combining the relations (35) and (39), we obtain (24). Theorem 5 is proved. ►

In [1] were considered the unbounded domains Ω with compact boundaries, for which $\sigma_p(L) = \varnothing$. It was proved [1, Lemma 9.17] that for any bounded domain $\Omega' \subset \Omega$ the following relation holds:

$$\liminf_{t\to\infty} \mathcal{E}_{\Omega'}(t) = 0. \quad (40)$$

It is readily seen that (40) follows from (24).

Absolute Continuity of Spectrum and Decay of Local Energy. The spectrum $\sigma(L)$ of self-adjoint operator $L : D(L) \to H$ is absolutely continuous $\sigma(L) = \sigma_{ac}(L)$ if $(E(\lambda)h, h)$ is an absolutely continuous function of λ for all $h \in H$ [15].

Theorem 6. *Let $\sigma(L) = \sigma_{ac}(L)$. Then for all $f \in \overset{o}{H}{}^1(\Omega)$, $g \in L_2(\Omega)$ and all bounded domains $\Omega' \subset \Omega$*

$$\lim_{t\to\infty} \mathcal{E}_{\Omega'}(t) = 0. \tag{41}$$

◀ We can assume without loss of generality (as in the proof of Theorem 5) that $f, g \in \mathcal{D}(\Omega) \subset D(L^p)$ for all $p = 1, 2, \ldots$ So, we can prove (41) for this case and suppose that the inequalities (25) holds. Now, for an arbitrary function $q(x) \in \mathcal{D}(\Omega)$ we have the equality

$$(u_t, q)_{L_2(\Omega)} = -\int_0^\infty \sin(\sqrt{\lambda}t)\sqrt{\lambda}\, d(E(\lambda)f, q) + \\ + \int_0^\infty \cos(\sqrt{\lambda}t)\, d(E(\lambda)g, q). \tag{42}$$

It follows from $\sigma(L) = \sigma_{ac}(L)$ and (25) that $d(E(\lambda)f, q) = m_1(\lambda)\, d\lambda$, $d(E(\lambda)g, q) = m_2(\lambda)\, d\lambda$ where

$$\int_0^\infty \lambda^{2p} |m_1(\lambda)|\, d\lambda < \infty; \quad \int_0^\infty \lambda^{2p} |m_2(\lambda)|\, d\lambda < \infty \tag{43}$$

for $p = 0, 1, 2, \ldots$ Therefore, by (42) we have

$$(u_t, q)_{L_2(\Omega)} = -\int_0^\infty \sqrt{\lambda} \sin(\sqrt{\lambda}t) m_1(\lambda)\, d\lambda + \int_0^\infty \cos(\sqrt{\lambda}t) m_2(\lambda)\, d\lambda = \\ = -2\int_0^\infty \sin(zt) z^2 m_1(z^2)\, dz + 2\int_0^\infty \cos(zt) z m_2(z^2)\, dz. \tag{44}$$

It now follows from the inequality

$$\sqrt{\lambda} \le \frac{\lambda + 1}{2} \tag{45}$$

and (43) that

$$\int_0^\infty z^2 |m_1(z^2)|\, dz + \int_0^\infty z\, |m_2(z^2)|\, dz = \\ = \frac{1}{2}\int_0^\infty \sqrt{\lambda} |m_1(\lambda)|\, d\lambda + \frac{1}{2}\int_0^\infty |m_2(\lambda)|\, d\lambda < \infty.$$

Now, applying the Riemann – Lebesgue Lemma to (44), we obtain

$$\lim_{t\to\infty} (u_t, q)_{L_2(\Omega)} = 0. \tag{46}$$

Moreover, for $q(x) \in \mathcal{D}(\Omega)$ we have:

$$(u, q)_{\widetilde{H}} = (\nabla u, \nabla q)_{L_2(\Omega)} = (\sqrt{L}u, \sqrt{L}q)_{L_2(\Omega)} =$$
$$= -\int_0^\infty \lambda \cos(\sqrt{\lambda}t)\, d(E(\lambda)f, q) - \int_0^\infty \sqrt{\lambda}\sin(\sqrt{\lambda}t)\, d(E(\lambda)g, q) =$$
$$= 2\int_0^\infty \cos(zt) z^3 m_1(z^2)\, dz + 2\int_0^\infty z^2 \sin(zt) m_2(z^2)\, dz. \quad (47)$$

By (43) and (45) the following integrals are finite:

$$\int_0^\infty z^3 |m_1(z^2)|\, dz + \int_0^\infty z^2 |m_2(z^2)|\, dz =$$
$$= \frac{1}{2}\int_0^\infty \lambda |m_1(\lambda)|\, d\lambda + \frac{1}{2}\int_0^\infty \sqrt{\lambda}|m_2(\lambda)|\, d\lambda < \infty.$$

Therefore, applying the Riemann – Lebesgue Lemma to (47), we obtain

$$\lim_{t\to\infty} (u(t, x), q(x))_{\widetilde{H}} = 0. \quad (48)$$

The space $\mathcal{D}(\Omega)$ is dense in $\widetilde{H}$. After closure the relation (48) with respect to q we obtain (48) for all $q \in \widetilde{H}$. Let us prove that for any $R > 0$

$$\lim_{t\to\infty} \|u_t(t, x)\|_{L_2(\Omega_R)} = 0. \quad (49)$$

By (32) the set of functions $\{u_t(t, x)\}$, $t > 0$ is a compact set in $L_2(\Omega_R)$. Let $\{h_{j,R}(x)\}$, $j = 1, 2, \ldots$, be an orthonormal basis in $L_2(\Omega_R)$. By the compactness criterion [21], for any $\varepsilon > 0$ there exists $N > 0$ such that $u_t(t, x) = \sum\limits_{j=1}^{N} c_{j,R}(t) h_{j,R}(x) + \sum\limits_{j=N+1}^{\infty} c_{j,R}(t) h_{j,R}(x)$ for $t > 0$, $x \in \Omega_R$ and $\|\sum\limits_{j=N+1}^{\infty} c_{j,R} h_{j,R}\|_{L_2(\Omega_R)} < \varepsilon$ for all $t > 0$. Therefore,

$$\|u_t\|^2_{L_2(\Omega_R)} = \|\sum_{j=1}^{N} c_{j,R} h_{j,R}\|^2_{L_2(\Omega_R)} + \|\sum_{j=N+1}^{\infty} c_{j,R} h_{j,R}\|^2_{L_2(\Omega_R)} <$$
$$< \sum_{j=1}^{N} c^2_{j,R} + \varepsilon^2 = \sum_{j=1}^{N} (u_t, h_{j,R})^2_{L_2(\Omega)} + \varepsilon^2. \quad (50)$$

By the relation (46) $\lim\limits_{t\to\infty} (u_t, h_{j,R})_{L_2(\Omega)} = 0$ for $j = 1, 2, \ldots, N$. Apply (50) we conclude that for any $\varepsilon > 0$ $\limsup\limits_{t\to\infty} \|u_t\|^2_{L_2(\Omega_R)} \leq \varepsilon^2$. It means that

$$\lim_{t\to\infty} \|u_t\|_{L_2(\Omega_R)} = 0. \tag{51}$$

Let us prove now that for any $R > 0$ $\lim\limits_{t\to\infty} \|\nabla u\|_{L_2(\Omega_R)} = 0$. It now follows from (32) that the set of functions $\{u(t,x)\}$, $t > 0$, is a compact set in the space $\widetilde{H}_R$. Let $\{h_{j,R}(x)\}$, $j = 1, 2, \ldots$, be an orthonormal basis in $\widetilde{H}_R$. By the compactness criterion in the space $\widetilde{H}_R$ with basis $\{h_{j,R}\}$ for any $\varepsilon > 0$ there exists $N > 0$ such that $u(t,x) = \sum\limits_{j=1}^{N} b_{j,R}(t)h_{j,R}(x) + {}$ $+ \sum\limits_{j=N+1}^{\infty} b_{j,R}(t)h_{j,R}(x)$, and $\|\sum\limits_{j=N+1}^{\infty} b_{j,R}h_{j,R}\|_{\widetilde{H}_R} < \varepsilon$ for all $t > 0$. Therefore,

$$\|u\|^2_{\widetilde{H}_R} = \|\sum_{j=1}^{N} b_{j,R}h_{j,R}\|^2_{\widetilde{H}_R} + \|\sum_{j=N+1}^{\infty} b_{j,R}h_{j,R}\|^2_{\widetilde{H}_R} <$$

$$< \sum_{j=1}^{N} b^2_{j,R} + \varepsilon^2 = \sum_{j=1}^{N} (u, h_{j,R})^2_{\widetilde{H}_R} + \varepsilon^2 = \sum_{j=1}^{N} \left(u, \tilde{h}_{j,R}\right)^2_{\widetilde{H}} + \varepsilon^2, \tag{52}$$

where the functions $\tilde{h}_{j,R} \in \widetilde{H}$ satisfy the equality $\left(v, \tilde{h}_{j,R}\right)_{\widetilde{H}} = (v, h_{j,R})_{\widetilde{H}_R}$ for all $v \in \widetilde{H}$. By the relation (48) we obtain $\lim\limits_{t\to\infty} \left(u, \tilde{h}_{j,R}\right)_{\widetilde{H}} = 0$ for $j = 1, \ldots, N$. Apply the equality (52), we have for any $\varepsilon > 0$

$$\limsup_{t\to\infty} \|\nabla u\|^2_{L_2(\Omega_R)} = \limsup_{t\to\infty} \|u\|^2_{\widetilde{H}_R} \leq \sum_{j=1}^{N} \lim_{t\to\infty} \left(u, \tilde{h}_{j,R}\right)^2_{\widetilde{H}} + \varepsilon^2 = \varepsilon^2.$$

In other words,

$$\lim_{t\to\infty} \|\nabla u\|_{L_2(\Omega_R)} = 0. \tag{53}$$

Now, for any bounded $\Omega' \subset \Omega$ we take R sufficiently large such that $\Omega' \subset \Omega_R$. Finally, combining the relations (51) and (53), we get the equality (41). Proof of the Theorem 6 is complete. ►

Conclusion. The results explained in the previous sections show how the basic information about spectrum of the Laplace operator allows us to study the qualitative properties of solutions of the mixed problem to hyperbolic equation. To obtain the rate of decay for local energy function $\mathcal{E}_{\Omega'}(t)$ we need the estimates to resolvent function of the Laplace operator in the complex domain [2, 11–13].

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