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OPTIMAL REPRESENTATION OF MULTIVARIATE FUNCTIONS OR DATA IN VISUALIZABLE LOW-DIMENSIONAL SPACES[1]

It is intended to find the best representation of high-dimensional functions or multivariate data in the $L_2(\Omega)$ space with the fewest number of terms, each of them is a combination of one-variable function. A system of non-linear integral equations has been derived as an eigenvalue problem of gradient operator in the above-said space. It is proved that the complete set of eigenfunctions generated by the gradient operator constitutes an orthonormal system, and any function of $L_2(\Omega)$ can be expanded with the fewest terms and exponential rapidity of convergence. It is also proved as a Corollary, all eigenvalues of the integral operators has multiplicity equal to 1 if the dimension of the underlying space $\mathbb{R}^n$ is $n = 2$, 4 and 6.

The analysis and processing of massive amount of multivariate data or high-dimensional functions have become a basic need in many areas of scientific exploration and engineering. To reduce the dimensionality for compact representation and visualization of high-dimensional information appear imperative in exploratory research and engineering modeling. Since D. Hilbert raised the 13^{th} problem in 1900, the study on possibility to express high-dimensional functions via composition of lower-dimensional functions has gained considerable success [1, 2]. Nonetheless, no methods of realization are ever indicated, and not even all integrable functions can be treated this way, *a fortiori* functions in $L_2(\Omega)$. The common practice is to expand high-dimensional functions into a convergent series in terms of a chosen orthonormal basis with lower dimensional ones. However, the length and rapidity of convergence of the expansion heavily depend upon the choice of basis. In this paper an attempt is made to seek an optimal basis for a given function provided with fewest terms and rapidest convergence. All elements of the optimal basis turned out to be products of single-variable functions taken from the unit balls of ingredient spaces. The proposed theorems and schemes may find wide applications in data processing, visualization, computing, engineering simulation and decoupling of nonlinear control systems. The facts established in the theorems may have their own theoretical interests.

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Due to the limitation of space we report here the primary Theorems with abridged proofs. The detailed proofs will be contained in a separate paper.

Let $F(x) = F(x^1, x^2, \ldots, x^n)$ be an arbitrarily given function defined on the unit cube Ω in $\mathbb{R}^n$, $F \in L_2(\Omega)$,

$$\int\limits_{\Omega} \left(F(x^1, x^2, ..., x^n)\right)^2 dx^1 dx^2 ... dx^n \leqslant M^2 < \infty, \tag{1}$$

where $x = (x^1, x^2, \ldots, x^n)$ is a point in $\mathbb{R}^n$. It is intended to find a set of one-variable functions $\varphi(x^\alpha)$ whose product $\varphi^1(x^1)\varphi^2(x^2)\ldots\varphi^n(x^n) \in L_2(\Omega)$ would best, or optimally, approximate $F(x)$ with the least-square deviation:

$$L = \int\limits_{\Omega} \left(F(x^1, x^2, ..., x^n) - \varphi^1(x^1)\varphi^2(x^2)\ldots\varphi^n(x^n)\right)^2 d\Omega = \min, \tag{2}$$

where $d\Omega = dx^1 dx^2 \ldots dx^n$.

Suppose each $\psi^\alpha(x^\alpha)$ is taken from the unit balls $B_\alpha \in L_2^{(\alpha)}(0,1)$, $B_\alpha = \{\|\psi^\alpha\|_{L_2(0,1)} \leq 1\}$, $\alpha = 1, 2, \ldots, n$, the above requirement (2) can be rewritten as

$$L = \inf_{\psi^\alpha \in B_\alpha} \int\limits_{\Omega} (F(x) - \lambda \prod_{\alpha=1}^{n} \psi^\alpha(x^\alpha))^2 d\Omega. \tag{3}$$

Opening up the brackets on the right side we have

$$\int\limits_{\Omega} (F(x) - \lambda \prod_{\alpha=1}^{n} \psi^\alpha(x^\alpha))^2 d\Omega =$$
$$= \int\limits_{\Omega} (F^2(x) - 2\lambda F(x) \prod_{\alpha=1}^{n} \psi^\alpha(x^\alpha) + \lambda^2 \prod_{\alpha=1}^{n} (\psi^\alpha(x^\alpha))^2) d\Omega.$$

It is easy to verify that (3) holds if and only if there exists a product of n functions $\prod\limits_{\alpha} \varphi^\alpha$, $\varphi^\alpha \in B_\alpha$, and a real $\lambda \in \mathbb{R}$, $\lambda \neq 0$, which enable the following functional to achieve supremum on all unit balls B_α,

$$\lambda = \sup_{\psi^\alpha \in B_\alpha} \int\limits_{\Omega} F(x) \prod_{\alpha=1}^{n} \psi^\alpha(x^\alpha) d\Omega =$$
$$= \int\limits_{\Omega} F(x^1, x^2, \ldots, x^n)\varphi^1(x^1)\varphi^2(x^2)\ldots\varphi^n(x^n) dx^1 dx^2 \ldots dx^n. \tag{4}$$

For the convenience of discussion in the sequel we distinguish the same spaces $L_2^{(\alpha)}(0,1)$, $\alpha = 1,2,\ldots,n$, and construct a product space $L_2^n(0,1) = L_2^{(1)}(0,1) \times L_2^{(2)}(0,1) \times \ldots \times L_2^{(n)}(0,1)$, $L_2^n(0,1) = = \left\{\sum_\beta a_\beta \prod_{\alpha=1}^n \psi_\beta^\alpha,\ \psi_\beta^\alpha \in L_2^{(\alpha)}(0,1),\ \alpha = 1, 2, \ldots, n,\ \beta = 1, 2, \ldots, N\right\}$.

After introduction of inner product for $\psi, \varphi \in L_2^n(0,1)$, $\psi = \prod_{\alpha=1}^n \psi^\alpha$, $\varphi = \prod_{\alpha=1}^n \varphi^\alpha$,

$$\langle \psi, \varphi \rangle = \left\langle \prod_{\alpha=1}^n \psi^\alpha, \prod_{\alpha=1}^n \varphi^\alpha \right\rangle = \prod_{\alpha=1}^n \langle \psi^\alpha, \varphi^\alpha \rangle = \prod_{\alpha=1}^n \int_0^1 \psi^\alpha(x^\alpha)\varphi^\alpha(x^\alpha)dx^\alpha,$$

with induced natural norm

$$\left\| \prod_{\alpha=1}^n \psi^\alpha \right\|_{L_2^n(0,1)} = \prod_{\alpha=1}^n \|\psi^\alpha\|_{L_2^{(\alpha)}(0,1)},$$

$L_2^n(0,1)$ becomes a linear normed space. The $L_2^n(0,1)$ defined above can be embedded into $L_2(\Omega)$ with preserved norm and becomes a dense subset of the latter [3, 4], while $L_2^{(\alpha)}(0,1)$ is a closed subspace of $L_2(\Omega)$, since $\|\psi^\alpha\|_{L_2^{(\alpha)}(0,1)}^{=} \|\psi^\alpha\|_{L_2(\Omega)}$ always holds on Ω the other hand, for the multilinear functional $f : L_2^n(0,1) \to R$, defined by

$$f\left(\prod_{\alpha=1}^n \psi^\alpha\right) = \int_\Omega F(x) \prod_\alpha \psi^\alpha(x^\alpha) d\Omega, \tag{5}$$

the following inequality holds for every element of $L_2^n(0,1)$:

$$\left| f\left(\prod_\alpha \psi^\alpha\right) \right| \leq M \prod_\alpha \|\psi^\alpha\|_{L_2^{(\alpha)}(0,1)},$$

where M is the lower bound defined in (1). Hence f is bounded and, by Banach-Steinhaus theorem, is totally continuous in $L_2^n(0,1)$ [5].

Let B_α be the unit ball of $L_2^{(\alpha)}(0,1)$, $B_\alpha = \{\psi^\alpha \in L_2^{(\alpha)}(0,1), \|\psi^\alpha\| \leq 1\}$, and $B^n = B_1 \times B_2 \times \cdots B_n$. First of all we need the following Lemma.

Lemma 1. The n-linear form (5) can achieve its supremum on B^n. Whenever $F(x) \neq 0$, the supremum λ is positive,

$$\lambda = \sup_{\psi^\alpha \in B_\alpha} \int_\Omega F(x) \prod_\alpha \psi^\alpha d\Omega > 0. \tag{6}$$

Proof. Since all unit balls B_α, $\alpha = 1, 2, \ldots, n$, are weak compact, by the Banach-Alaoglu theorem, every sequence $\{\psi_k^\alpha\}$ enabling (4) to approach supremum contains a subsequence weakly converging to some element φ^α in B_α [5, 6]. Now we show that there exists a sequence $\prod\limits_{\alpha=1}^{n} \psi_k^\alpha$, $k = 1, 2, \ldots$, that converges weakly to an element $\prod\limits_{\alpha=1}^{n} \varphi^\alpha \in L_2^n(0, 1)$ at which the following functional achieves its supremum,

$$\lim_{k\to\infty} \int_\Omega F(x^1, \cdots, x^n) \prod_\alpha \psi_k^\alpha d\Omega =$$

$$= \int\limits_\Omega F(x^1, \cdots, x^n) \prod_\alpha \varphi^\alpha d\Omega = \lambda = \sup_{\psi^\alpha \in B_\alpha} \int\limits_\Omega F(x) \prod_\alpha \psi^\alpha(x^\alpha) d\Omega,$$

or

$$\lim_{k\to\infty} \int_\Omega F(x^1, \cdots, x^n) [\prod_\alpha \psi_k^\alpha - \prod_\alpha \varphi^\alpha] d\Omega = 0.$$

Due to the identity

$$\prod_\alpha \psi_k^\alpha - \prod_\alpha \varphi^\alpha =$$

$$= \prod_\alpha \psi_k^\alpha - \varphi^1 \psi_k^2 \cdots \psi_k^n + \varphi^1 \psi_k^2 \cdots \psi_k^n - \varphi^1 \varphi^2 \psi_k^3 \cdots \psi_k^n + \cdots$$

$$+ \varphi^1 \varphi^2 \cdots \varphi^{n-1} \psi_k^n - \prod_\alpha \varphi^\alpha,$$

we have

$$\int\limits_\Omega F \left[\prod_\alpha \psi_k^\alpha - \prod_\alpha \varphi^\alpha \right] d\Omega =$$

$$= \sum_{i=1}^n \int\limits_{\Omega_i} \left[\int\limits_0^1 F \cdot (\psi_k^i - \varphi^i) dx^i \right] \prod_{\alpha=1}^{i-1} \varphi^\alpha \prod_{\beta=i+1}^{n} \psi_k^\beta d\Omega_i =$$

$$= \sum_{i=1}^n \int\limits_{\Omega_i} F_k \prod_{\alpha=1}^{i-1} \varphi^\alpha \cdot \prod_{\beta=i+1}^{n} \psi_k^\beta d\Omega_i,$$

where $d\Omega_i = dx^1 \cdots dx^{\hat{i}} \cdots dx^n$; $\hat{i}$ means without ith coordinate, and

$$F_k(x^1, \cdots, x^{\hat{i}}, \cdots, x^n) = \int\limits_0^1 F(x^1, \cdots, x^n) [\psi_k^i(x^i) - \varphi^i(x^i)]\, dx^i.$$

By the weak compactness of B_α, for any fixed point $(x^1, \cdots, x^{\hat{i}}, \cdots, x^n)$ in Ω_i, $F_k(x^1, \cdots, x^{\hat{i}}, \cdots, x^n) \to 0$ as $(\psi^i_k(x^i) - \varphi^i(x^i))$ tends weakly to 0. By the Dominated Convergence Theorem [7], $\|F_k\|_{L_2(\Omega_i)} \to 0$, k= 1, 2, $\ldots$, n, hence

$$\left| \int\limits_\Omega F(x^1, \cdots, x^n) \left[\prod_\alpha \psi^\alpha_k - \prod_\alpha \varphi^\alpha \right] d\Omega \right| \leq \sum_{i=1}^n \|F_k\|_{L_2(\Omega_i)} \to 0$$

as $k \to \infty$. This is to be shown for the first part of the Lemma.

To verify the second statement we take an orthonormal basis $\Big\{ e^\beta_\alpha(x^\alpha)$, $\beta = 1, 2, \cdots \Big\}$ in each $L_2^{(\alpha)}(0,1)$ and construct a set E, $E = \Big\{ \prod\limits_{\alpha=1}^n e^{\gamma_\alpha}_\alpha(x^\alpha)$, $\gamma_\alpha = 1, 2, \cdots \Big\}$, each γ_α runs over $\mathbb{N}$ independently. Since $L_2^n(0,1)$ is dense in $L_2(\Omega)$, E becomes an orthonormal basis of the latter [4, 8]. Any element $F \in L_2(\Omega)$ can be expressed uniquely in the form of Fourier series,

$$F(x^1, \cdots, x^n) = \sum_{\beta=1}^\infty P_\beta \prod_{\alpha=1}^n e^{\beta_\alpha}_\alpha, \quad P_\beta = \left\langle F, \prod_\alpha e^{\beta_\alpha}_\alpha \right\rangle_{L_2(\Omega)}.$$

By assumption $F \neq 0$, there must be some $P_k \neq 0$. Let $\psi_k = (\text{sign}\, P_k) \prod\limits_\alpha e^k_\alpha$. Substitute the above series into (6), and take inner product with just defined ψ_k , we obtain immediately $\lambda > |P_k| > 0$, what is claimed in the Lemma.

Now we proceed to establish the necessary conditions which a solution of (4), $\prod\limits_\alpha \varphi^\alpha$, should satisfy. Suppose the expression (4) achieves its supremum at some element $\prod\limits_\alpha \varphi^\alpha \in B^n$. According to Lagrange Principle, $\prod\limits_\alpha \varphi^\alpha$ must satisfy the following conditions with a multiplier λ' [5, 9]:

$$D\left(\int\limits_\Omega F(x^1, \cdots, x^n) \prod_\alpha \varphi^\alpha dx - \lambda' \left(\left\langle \prod_\alpha \varphi^\alpha, \prod_\alpha \varphi^\alpha \right\rangle_{L_2(\Omega)} - 1 \right) \right) = 0, \tag{7}$$

where D denotes the Gateaux directional derivative with respect to all $\varphi\alpha$, λ' is a real to be determined. According to the rules of differentiation, $Df = \sum\limits_{i=1}^n D_i f$, $D_i f$ is the partial derivative with respect to φ^i. Let h^i

be arbitrary element in $L_2^{(i)}(0,\,1)$, $t\in[0,1]$, $\lambda = 2\lambda'$. A straightforward computation yields

$$D_i f = \lim_{t\to 0}\frac{1}{t}\left[f\left(\prod_{\alpha\neq i}\varphi^\alpha(\varphi^i + th^i\right) - \right.$$

$$\left. - f\left(\prod_{\alpha=1}^{n}\varphi^\alpha\right)\right] = \left\langle \Phi_i\left(\prod_{\alpha\neq i}\varphi^\alpha\right) - \lambda\varphi^i,\ h^i\right\rangle_{L_2^{(i)}(0,1)} = 0.$$

Due to the arbitrariness of h^i, we have

$$\Phi_i\left(\prod_{\alpha\neq i}\varphi^\alpha\right) - \lambda\varphi^i = 0, \quad i = 1, 2, \cdots n, \tag{8}$$

or, after unfolding,

$$\lambda\varphi^i(x^i) = \Phi_i\left(\prod_{\alpha\neq i}\varphi^\alpha\right) =$$

$$= \int\limits_{\Omega_i} F(x^1, \cdots, x^n)\varphi^1(x^1)\cdots\varphi^{\hat{i}}(x^i)\cdots\varphi^n(x^n)dx^1\cdots$$

$$\cdots dx^{\hat{i}}\cdots dx^n, \quad i = 1,\, 2, \cdots n, \quad \text{(8')}$$

where $\hat{i}$ means absence of ith coordinate, Ω_i is the $(n-1)$ dimensional unit cube without x^i.

The operator $\Phi = (\Phi_1,\,\Phi_2,\ldots,\Phi_n)$ generated by G-derivative and defined in (8') is called the gradient operator of functional (5). Now we proceed to prove the following, the primary theorem as a start-point for further investigation.

Theorem 2. For any given $F(x)\in L_2(\Omega)$, $F(x)\neq 0$, its gradient operator Φ or, equivalently, the system of homogeneous integral equations (8') possesses at least one positive eigenvalue. The greatest eigenvalue and its associated eigenfunctions satisfy (4), and are a solution of this supremum problem.

Proof (abridged). It is known that for any given $F(x)\in L_2(\Omega)$, all components of its gradient operator Φ_i, defined by (8), are compact [6, 7], so the range $\Phi_i(B_i^{n-1})$ is a compact subset in $L_2^{(i)}(0,\,1)$, here $B^{n-1} = \left\{\prod\limits_{\alpha\neq i}\psi^\alpha(x^\alpha),\ \|\psi^\alpha\|\leq 1\right\}$. By Lemma 1, there exists a sequence in B^n, $\left\{\prod\psi_k^\alpha\right\}$, weakly convergent to $\prod\limits_\alpha\varphi^\alpha\in B^n$ as $k\to\infty$ so that the following holds,

$$\lim_{k\to\infty}\left\langle \Phi_i\left(\prod_{\alpha\neq i}\psi_k^\alpha\right),\ \psi_k^i\right\rangle_{L_2(0,1)} =$$

$$= \left\langle \Phi_i\left(\prod_{\alpha\neq i}\varphi^\alpha\right),\ \varphi^i\right\rangle_{L_2(0,1)} = \lambda,\ i=1,2,\cdots,n.$$

Now we construct new functions,

$$\eta_k^i(x^i) = \lambda\psi_k^i - \Phi_i\left(\prod_{\alpha\neq i}\psi_k^\alpha\right) =$$

$$= \lambda\psi_k^i(x^i) - \int_{\Omega_i} F(x^1,x^2,\cdots,x^n)\psi_k^1(x^1)\cdots$$

$$\cdots\psi_k^{\hat{i}}\cdots\psi_k^n(x^n)dx^1\cdots dx^{\hat{i}}\cdots dx^n,\ \|\psi_k^\alpha\|\leq 1,\ i=1,2,\cdots,n.$$

An accurate calculation of the norm of η_k^i shows that Lemma 1 implies also while $\prod\limits_\alpha\psi_k^\alpha$ approaches weakly to its limit $\prod\limits_\alpha\varphi^\alpha$, η_k^i strongly tends to zero. Namely,

$$\lim_{k\to\infty}\langle\eta_k^i,\eta_k^i\rangle = \lim_{k\to\infty}\left\|\lambda\psi_k^i - \Phi_i(\prod_{\alpha\neq i}\psi_k^\alpha)\right\|^2 = 0, \quad i=1,2,\cdots,n.$$

This indicates, the sequence $\{\psi_k^\alpha,\ \alpha=1,2,\cdots,n,\ k=1,2,\cdots\}$ has a strong limit, denoted again by $\{\varphi^\alpha,\ \alpha=1,\ 2,\ldots,n\}$, which satisfies (8). Taking inner product for (8') with φ^i, we obtain finally,

$$\left\langle \Phi_i\left(\prod_{\substack{\alpha=1\\ \alpha\neq i}}\varphi^\alpha\right),\varphi^i\right\rangle =$$

$$= \int_\Omega F(x^1,x^2,\cdots,x^n)\varphi^1(x^1)\varphi^2(x^2)\cdots\varphi^n(x^n)dx^1dx^2\cdots dx^n = \lambda, \quad (9)$$

that is, $\prod\limits_\alpha\varphi^\alpha$ is a solution of (8), as claimed in the Theorem.

Let λ_1 and $\prod\limits_{\alpha=1}^n\varphi_1^\alpha$ are the greatest eigenvalue and its associated eigenfunctions, respectively. Construct new function

$$F_1(x^1,x^2,\cdots,x^n) = F_0(x^1,x^2,\cdots,x^n) - \lambda_1\prod_{\alpha=1}^n\varphi_1^\alpha(x^\alpha).$$

Let $F_1 \neq 0$. It is easy to check that the norm of F_1in $\mathrm{L}_2(\Omega)$ can be calculated as

$$\|F_1\|^2 = \left\langle F_0 - \lambda_1 \prod \varphi_1^\alpha, F_0 - \lambda_1 \prod \varphi_1^\alpha \right\rangle =$$

$$= \|F_0\|^2 - 2\lambda_1 \left\langle F_0, \prod \varphi_1^\alpha \right\rangle + \lambda_1^2 \left\langle \prod \varphi_1^\alpha, \prod \varphi_1^\alpha \right\rangle = \|F_0\|^2 - \lambda_1^2.$$

So long as $F_1 \neq 0$, the Lemma 1 and Theorem 2 are applicable for F_1 as well. Having F_0 replaced by F_1 in (8), one gets a new operator Φ_1 and, correspondingly, new system of equations (8').

By Theorem 2, it possesses at least one positive eigenvalue λ_2 and associated eigenfunctions $\prod\limits_\alpha \varphi_2^\alpha$ of (8) with F replaced by F_1. Similarly, we have

$$F_2 = F_1 - \lambda_2 \prod_{\alpha=1}^n \varphi_2^\alpha = F_0 - \lambda_1 \prod_{\alpha=1}^n \varphi_1^\alpha - \lambda_2 \prod_{\alpha=1}^n \varphi_2^\alpha,$$

with its norm

$$\|F_2\|^2 = \|F_1\|^2 - \lambda_2^2 = \|F_0\|^2 - \lambda_1^2 - \lambda_2^2.$$

The process can be continued inductively, if $F_N \neq 0$,

$$F_N = F_{N-1} - \lambda_N \prod_{\alpha=1}^n \varphi_N^\alpha = F_0 - \sum_{\beta=1}^N \lambda_\beta \prod_{\alpha=1}^n \varphi_\beta^\alpha, \tag{10}$$

and

$$\|F_N\|^2 = \|F_{N-1}\|^2 - \lambda_N^2 = \|F_0\|^2 - \sum_{\beta=1}^N \lambda_\beta^2. \tag{11}$$

Further, each F_N generates its own gradient operator $\Phi_N = (\Phi_{N1}, \ldots, \Phi_{Nn})$,

$$\Phi_{Ni}\left(\prod_{\alpha\neq i} \varphi_{N+1}^\alpha\right) = \int_{\Omega_i} F_N(x) \prod_{\alpha\neq i} \varphi_{N+1}^\alpha(x^\alpha) d\Omega_i = \lambda_{N+1} \varphi_{N+1}^i(x^i),$$

$$i = 1, 2, \cdots, n. \tag{12}$$

If the process continues infinitely, (10) becomes an infinite series. Now we prove that the following equality holds as $N \to \infty$ in the norm of $L_2(\Omega)$,

$$F_0(x^1, x^2, \cdots, x^n) = \sum_{\beta=1}^\infty \lambda_\beta \prod_{\alpha=1}^n \varphi_\beta^\alpha(x^\alpha). \tag{13}$$

Theorem 3. Any given $F(x) \in L_2(\Omega)$, $F \neq 0$, can be expressed in the form of series (13) with all positive eignevalues and associated eignefunctions generated by the sequence of gradient operators $\{\Phi_\beta\}$. The series (13) converges exponentially to $F(x)$ in the norm of $L_2(\Omega)$.

Proof. If the process of construction described in (10) terminates at a finite step N, the validity of the first part of Theorem is apparent. Now suppose that (10) becomes an infinite series (13) when $N \to \infty$. It is obvious from (10), however large is N, one always has $\|F_N\|^2 \geq 0$, and

$$\sum_{\beta=1}^{N} \lambda_\beta^2 \leqslant \|F_0\|^2 . \tag{14}$$

The necessary condition of convergence for the series on the left side is $\lambda_N \to 0$ as $N \to \infty$, and (6) implies that the following relationship holds uniformly on B^n,

$$\lambda_{N+1} = \sup_{\psi^\alpha \in B_\alpha} \int_\Omega F_N(x) \prod_\alpha \psi^\alpha(x^\alpha) d\Omega = \left\langle F_N(x), \prod_\alpha \varphi_{N+1}^\alpha \right\rangle_{L_2(\Omega)} \geq$$
$$\geq \left| \left\langle F_N(x), \prod_\alpha \psi^\alpha \right\rangle_{L_2(\Omega)} \right|, \ \forall \psi^\alpha \in B_\alpha ,$$

here $\prod\limits_\alpha \varphi_{N+1}^\alpha$ is a solution of (6) and (8) with $F_0(x)$ replaced by $F_N(x)$. Therefore, when $N \to \infty$,

$$\lim_{N\to\infty} \left\langle F_N(x), \prod_\alpha \psi^\alpha \right\rangle_{L_2(\Omega)} \to 0, \ \forall \psi^\alpha \in B_\alpha, \ \alpha = 1, 2, \cdots, n.$$

It is evident that B^n is a fundamental set, i.e., it spans $L_2^n(0,\, 1)$, is a dense subset of $L_2(\Omega)$. The above condition suffices for F_N to converge weakly-star to some element F_∞ which is equivalent to 0 in the weak topology [4],

$$\lim_{N\to\infty} \sup_{B^n} \int_\Omega F_N(x) \prod_\alpha \psi^\alpha(x^\alpha) d\Omega = \sup_{B^n} \left\langle F_\infty, \prod_\alpha \psi^\alpha \right\rangle_{L_2(\Omega)} = 0.$$

Recall the fact that the set $E = \left\{ \prod\limits_\alpha e_\alpha^{\gamma_\alpha}, \ \gamma_\alpha \in \mathbb{N} \right\}$ consisting of combinations of orthonormal bases of $L_2^{(\alpha)}(0,\, 1)$ is a complete orthonormal basis for $L_2(\Omega)$. Notice$E \subset B^n$, by Lemma 1, it implies

$$0 = \sup_{B^n} \left\langle F_\infty, \prod_\alpha \psi^\alpha \right\rangle \geq \sup_E \left\langle F_\infty, \prod_\alpha e_\alpha^{\gamma_\alpha} \right\rangle \geqslant 0.$$

Which means all Fourier coefficients of F_∞ are zero, hence $F_\infty = 0$. Then (13) holds in the sense

$$\|F_0\|^2 = \sum_{\beta=1}^{\infty} \lambda_\beta^2. \tag{15}$$

Now we proceed to justify the second statement claimed in the Theorem about the convergence rapidity of the expansion (15). By assumption all $F_\beta \neq 0$, and due to (10) and (11), the following relations and the continuous multiplication are well defined,

$$\frac{\|F_1\|^2}{\|F_0\|^2} = \frac{\|F_0\|^2 - \lambda_1^2}{\|F_0\|^2} = 1 - \frac{\lambda_1^2}{\|F_0\|^2}, \cdots$$

$$\cdots, \frac{\|F_N\|^2}{\|F_{N-1}\|^2} = \frac{\|F_{N-1}\|^2 - \lambda_N^2}{\|F_{N-1}\|^2} = 1 - \frac{\lambda_N^2}{\|F_{N-1}\|^2},$$

$$\frac{\|F_N\|^2}{\|F_0\|^2} = \frac{\|F_N\|^2}{\|F_{N-1}\|^2} \frac{\|F_{N-1}\|^2}{\|F_{N-2}\|^2} \cdots \frac{\|F_2\|^2}{\|F_1\|^2} \frac{\|F_1\|^2}{\|F_0\|^2} = \prod_{\beta=1}^{N} \left(1 - \frac{\lambda_\beta^2}{\|F_{\beta-1}\|^2}\right).$$

Denote $\alpha_\beta^2 = \dfrac{\lambda_\beta^2}{\|F_{\beta-1}\|^2}$, namely $\lambda_\beta = \alpha_\beta \|F_{\beta-1}\|$. By the proved previously, $0 < \alpha_\beta \leq 1$, and notice that the inequality $(1-\alpha) \leq e^{-\alpha}$ always holds, then we have

$$\|F_N\|^2 = \|F_0\|^2 \prod_{\beta=1}^{N} (1 - \alpha_\beta^2) \leqslant \|F_0\|^2 e^{-\sum\limits_{\beta=1}^{N} \alpha_\beta^2},$$

or

$$\|F_N\| \leq \|F_0\| e^{-\frac{1}{2}\sum\limits_{\beta=1}^{N} \alpha_\beta^2}. \tag{16}$$

Let $R_{N+1} = \|F_N\|^2 = \sum\limits_{\beta=N+1}^{\infty} \lambda_\beta^2$ be the sum of residual part of (15). It is easy to show the sum

$$\sum_{k=1}^{N} \alpha_k^2 = \sum_{k=1}^{N} \frac{\lambda_k^2}{R_k}$$

diverges to infinity as $N \to \infty$ [10], and the right side of (16) tends to zero exponentially as claimed in the Theorem. The unconditionality of convergence of (13) will be provided by Proposition 4.

In the above discussion we did not touch upon the properties of the set of eigenfunctions. We will see below, it is quite similar to the case of symmetrical integral operators, the set of all eigenfuctions generated by (12) constitutes an orthonormal system in $L_2(\Omega)$.

Proposition 4. For arbitrarily given $F(x) \in L_2(\Omega)$, $F \neq 0$, the set of all eigenfunctions $\left\{\prod_\alpha \varphi_\beta^\alpha,\ \beta = 1, 2, \ldots\right\}$ of the sequence of gradient operators $\Phi_\beta = (\Phi_{\beta 1}, \ldots, \Phi_{\beta n})$, defined by (12), constitutes an orthonormal system as an ingredient part of some complete orthonormal basis of $L_2(\Omega)$.

Proof. By definition of F_N, the identities $\left\langle F_\beta, \prod_\alpha \varphi_\beta^\alpha \right\rangle = 0$ hold for all $\beta = 1, 2, \ldots, N$. Each φ_{N+1}^α can be decomposed uniquely as $\varphi_{N+1}^\alpha = a_\alpha \varphi_N^\alpha + b_\alpha \bar\varphi_{N+1}^\alpha$, $\varphi_N^\alpha \perp \bar\varphi_{N+1}^\alpha$, and $a\alpha$, $b\alpha$ be constants of normalization. A substitution for φ_{N+1}^α yields

$$\prod_\alpha \varphi_{N+1}^\alpha =$$

$$= C_0 \prod_\alpha \varphi_N^\alpha + C_1 \sum_{i=1}^n \bar\varphi_{N+1}^i \prod_{\alpha \neq i} \varphi_N^\alpha + \cdots + C_n \prod_\alpha \bar\varphi_{N+1}^\alpha = C_0 \prod_\alpha \varphi_N^\alpha + P_{N+1}.$$

Clearly, $P_{N+1} \perp \prod_\alpha \varphi_N^\alpha$. Thus, due to the identities said above, $\lambda_{N+1} = \left\langle F_N, \prod_\alpha \varphi_{N+1}^\alpha \right\rangle = C_0 \left\langle F_N, \prod_\alpha \varphi_N^\alpha \right\rangle + \langle F_N, P_{N+1} \rangle = = \langle F_{N-1}, P_{N+1} \rangle$. It follows $\prod_\alpha \varphi_N^\alpha \perp \prod_\alpha \varphi_{N+1}^\alpha$. Similar analysis of φ_β^α for $\beta = N-1, \ldots, 1$ in succession, one obtains

$$\lambda_{N+1} = \sup_{\psi^\alpha \in B_\alpha} \int_\Omega F_N(x) \prod \psi^\alpha d\Omega =$$
$$= \sup_{\substack{\psi^\alpha \in B_\alpha \\ \psi^\alpha \in (L_N^{(\alpha)})^\perp}} \int_\Omega F_0(x) \prod_\alpha \psi^\alpha d\Omega = \int_\Omega F_0(x) \prod_\alpha \varphi_{N+1}^\alpha d\Omega. \quad (17)$$

This being true for all $N \in \mathbb{N}$ follows the set of eigenfunctions $\left\{\prod_\alpha \varphi_\beta^\alpha,\ \beta = 1, 2, \ldots\right\}$ constitutes a orthonormal system in $L_2(\Omega)$. Since for any orthonormal set S in a Hilbert space there is a complete orthonormal basis that contains S as its subset [11]. The Proposition is thus justified.

Remark. It appears the remarkable maximum property of gradient operators expressed in (17), which is entirely analogous with compact self-adjoint operators in Hilbert spaces. The Proposition also shows there are as many different orthonormal bases as the cardinality of different elements in $L_2(\Omega)$.

Corollary 5. If the dimension of underlying space $\mathbb{R}^n$ with $n = 2$, 4 and 6, all eigenvalues of gradient operators defined by (11) for arbitrarily given $F(x) \in L_2(\Omega)$, $F(x) \neq 0$, have multiplicity no more than 1.

Proof (abridged). Suppose the contrary, if there exist two different eigenfunctions $\varphi_1 = \prod\limits_\alpha \varphi^\alpha_{N+1,1}$ and $\varphi_2 = \prod\limits_\alpha \varphi^\alpha_{N+1,2}$ corresponding to the same eigenvalue λ_{N+1}, which enable the following functional to achieve its supremum on B^n,

$$\lambda_{N+1} = \sup_{\psi^\alpha \in B_\alpha} f_N(\prod_\alpha \psi^\alpha) = \\ = \sup_{\psi^\alpha \in B_\alpha} \int_\Omega F_N(x) \prod_\alpha \psi^\alpha d\Omega = f_N(\prod_\alpha \varphi^\alpha_{N+1,k}),\ k = 1, 2. \quad (18)$$

Now construct a new element $\varphi_3 = \prod\limits_\alpha (t\varphi^\alpha_1 + (1-t)\varphi^\alpha_2)$, $t \in [0,\,1]$, and put it into (18). After exposing the product, we have

$$f_N(\varphi_3, t) = f_N\left(\prod_\alpha (t\varphi^\alpha_1 + (1-t)\varphi^\alpha_2)\right) = \\ = t^n f_N\left(\prod_\alpha \varphi^\alpha_1\right) + t^{n-1}(1-t) f_N\left(\sum_{i=1}^n \varphi^i_2 \prod_{\hat{i}} \varphi^\alpha_1\right) + \\ + t^{n-2}(1-t)^2 f_N\left(\sum_{i\neq j} \varphi^i_2 \varphi^j_2 \prod_{\hat{i}\hat{j}} \varphi^\alpha_1\right) + \ldots \\ \ldots + t^2(1-t)^{n-2} f_N\left(\sum_{i\neq j} \varphi^i_1 \varphi^j_1 \prod_{\varphi^\alpha_2}\right) \\ + t(1-t)^{n-1} f_N\left(\sum_{i=1}^n \varphi^i_1 \prod_{\hat{i}} \varphi^\alpha_2\right) + (1-t)^n f_N\left(\prod_\alpha \varphi^\alpha_2\right). \quad (19)$$

By Proposition 4, φ^i_1 and φ^i_2 are mutually orthogonal. Dropping index N for brevity, we have the necessary condition,

$$Df = f\left(\sum_i h^i_1 \prod_{\hat{i}} \varphi^\alpha_1\right) = \sum_i \left\langle \varphi^i_1, h^i_1 \right\rangle \lambda = 0, \quad \forall h^i_1 \in L^{(i)}_2(0,1).$$

Take $h^i = \varphi_2^i$ in the above we see that the second term and the second from the end are zero. For computing the rest terms in the exposition (19) we invoke the sufficient conditions for f to achieve its maximum at φ. If the dimension of the underlying space, $\mathbb{R}^n$, n is even, then the following equations must be satisfied [9, 11],

$$D^k f\left(\prod_{\alpha} \varphi^\alpha\right) = 0, \qquad k = 1, 2, \cdots, n-1. \tag{20}$$

A direct calculation shows

$$D^2 f\left(\prod_{\alpha} \varphi_1^\alpha\right) = 2f\left(\sum_{i\neq j} h_1^i h_2^i \prod_{\hat{i}\hat{j}} \varphi_1^\alpha\right) - \sum_i \left\langle h_1^i, h_2^i\right\rangle \lambda = 0$$

for arbitrary h_1^i and h_2^i taken from $L_2^{(i)}(0,\, 1)$. Let $h_1^i = h_2^i = \varphi_2^i$. We get the value of the third term,

$$f_N\left(\sum_{i\neq j} \varphi_2^i \varphi_2^i \prod_{\hat{i}\hat{j}} \varphi_1^\alpha\right) = \frac{n}{2}\lambda_{N+1}.$$

The third term from the end of (19) is in complete symmetry with the above, and, due to (20), all the rest terms are zero except the first and last ones. Then,

$$f_N(\varphi_3,\ t) = \left(t^n + t^{n-2}(1-t)^2\frac{n}{2} + t^2(1-t)^{n-2}\frac{n}{2} + (1-t)^n\right)\lambda_{N+1}.$$

It follows from the assumption, $f_N(\varphi_3,\, 0) = f_N(\varphi_3, 1) = \lambda_{N+1}$, and it reaches minimum at $t = 1/2$. Let $t = 1/2$. It yields

$$f_N\left(\varphi_3,\ \frac{1}{2}\right) = f_N\left(\prod_{\alpha} \frac{\varphi_1^\alpha + \varphi_2^\alpha}{2}\right) = (2+n)2^{-n}\lambda_{N+1}.$$

Since $\|\varphi_1^\alpha + \varphi_2^\alpha\| = \sqrt{2}$, $\dfrac{\varphi_1^\alpha + \varphi_2^\alpha}{\sqrt{2}} \in B_\alpha$, we have

$$f_N\left(\varphi_3, \frac{1}{\sqrt{2}}\right) = f_N\left(\prod_{\alpha} \frac{\varphi_1^\alpha + \varphi_2^\alpha}{\sqrt{2}}\right) = (2+n)\cdot 2^{-\frac{n}{2}}\lambda_{N+1} = \rho(n)\lambda_{N+1}.$$

It is evident, if $\rho(n) \geq 1$ then $(\rho(n))^{\frac{1}{n}} \geq 1$, so that $\displaystyle\prod_{\alpha} \frac{\varphi_1^\alpha + \varphi_2^\alpha}{\sqrt{2}\rho^{\frac{1}{n}}} \in B^n$. A direct computation shows this is possible only for $n = 2$, 4 and 6. In these cases if $\varphi_1^{=} \displaystyle\prod_{\alpha} \varphi_{N+1,1}^\alpha$, and $\varphi_2^{=} \displaystyle\prod_{\alpha} \varphi_{N+1,2}^\alpha$ both render the supremum

λ_{N+1} to the functional f_N, then along the direction of the middle point φ_3 of segment joining φ_1^α and φ_2^α, also provides the supremum λ_{N+1} to f_N. By the assumption $\varphi_1 \neq \varphi_2$, there must be infinite amount of different elements in B^n at each of them f_N attains its supremum. This contradicts the compactness of gradient operators. That is, for $n = 2$, 4, and 6, the multiplicity of any eigenvalue of (12) is no more than 1. This completes the proof of the corollary.

The case $n = 2$ may cause particular theoretical interest [12,13]. Let $F(x, y)$ be defined on the unit rectangle B^2 of the plane and be square-integrable. By Theorem 2, it generates a gradient operator Φ, and (8) is reduced to

$$\Phi\psi = \int_0^1 F(x,y)\psi(y)dy = \lambda\varphi, \qquad \Phi^*\varphi = \int_0^1 F(x,y)\varphi(x)dx = \lambda\psi. \quad (21)$$

Apparently, φ and ψ are eigenfunctions of self-adjoint operators

$$\Phi\Phi^*\varphi = \lambda^2\varphi, \qquad \Phi^*\Phi\psi = \lambda^2\psi.$$

Corollary 5 claims for this case that all eigenvalues of Φ have multiplicity no more than 1. Indeed, suppose the contrary. Let $\varphi_1(x)\psi_1(y)$ and $\varphi_2(x)\psi_2(y)$ provide the same supremum λ_1 on B^2,

$$\lambda_1 = \sup_{\varphi\psi\in B^2} \int_\Omega F(x,y)\varphi(x)\psi(y)dxdy =$$

$$= \int_\Omega F(x,y)\varphi_k(x)\psi_k(y)dxdy, \qquad k = 1, 2. \quad (22)$$

By Proposition 4, $\varphi_1 \perp \varphi_2$ and $\psi_1 \perp \psi_2$. Let

$$\varphi_3\psi_3 = (t\varphi_1 + (1-t)\varphi_2)(t\psi_1 + (1-t)\psi_2), \quad 0 \leqslant t \leqslant 1.$$

It is easy to check,

$$\int_\Omega F(x,y)\varphi_3(x)\psi_3(y)dxdy = (1-2t+2t^2)\lambda_1, \quad \|\varphi_3\psi_3\| = 1-2t+2t^2,$$

thus

$$\int_\Omega F(x,y)\frac{\varphi_3(x)\psi_3(y)}{\|\varphi_3\psi_3\|}dxdy = \lambda_1, \ \ \forall t \in [0,1].$$

This means the functional reaches λ_1 on B^2 along all rays from the origin and intersecting any point of segments joining φ_1, φ_2 and ψ_1, ψ_2.

It contradicts the assumption and implies λ_1 is not supremum of the functional. One may notice, this is true if $F(x,y)$ replaced by $F_N(x,y) = $ $= F(x,y) - \sum_{\beta=1}^{N} \lambda_\beta \varphi_\beta \psi_\beta$ in (21). In particular, if $F(x,y) = F(y,x)$, it generates a self-adjoint operator, $\Phi = \Phi^*$, the above conclusion is also true for this special case of the fact described above.

From the geometrical point of view, it is known that the unit ball B of $L_2(\Omega)$ is strictly and uniformly convex [14,15]. It is believed that the $B \cap B^n$ possesses the same property. The equations (18) create a supporting tangent hyperplane to B^n in $L_2(\Omega)$. A conjecture arises that the claim made in Corollary 5 would be true for any finite dimensional underlying spaces $\mathbb{R}^n$. But we have had direct proof only for n even and $n \leq 6$. So the general question remains still open.

One may wonder what is the condition to be imposed on $F(x)$ for guaranteeing the convergence of the series (13) in space $L_1(\Omega)$ and in the Banach space of continuous functions $C(\Omega)$. The question arisen is that the assumption (1) is not enough to ensure the convergence of the infinite sum $\sum_{\beta} \lambda_\beta$ except $F(x)$ generates a nuclear gradient operators [16]. However, for our cases, according to the theories developed in [17, 18, 19], we can establish the following Theorem. We list it with the proof omitted.

Theorem 6. For any given function $F(x) \in L_2(\Omega)$, the series (13) converges uniformly in $L_1(\Omega)$. If $F(x)$ is continuous on Ω and possesses all continuous first partial derivatives in Ω, then the series of expansion (13) converges uniformly to the continuous function $F(x)$.

It is worth to re-emphasize, Theorem 3 and Proposition 4 have shown that for any high-dimensional square-integrable function $F(x)$ there exists an optimal orthonormal system of its own, consisting of eigenfunctions of its gradient operator, in terms of which $F(x)$ can be expanded with shortest length and rapidest convergence. Since each element of the system is a product of n single-variable functions, this may be a reliable way for reduction of dimensionality and compact expression of information contained in $F(x)$ in one-dimensional spaces. The inequality (16) provides *a posteriori* error estimate, in the process of computing the remaining error can be precisely estimated after completion of each step of calculation, this is thus a difference from *a priori* error estimate.

We recall that $L_2(\Omega)$ and l_2, the space of square-summable sequences of reals, are isometrically isomorphic. Each element of $L_2(\Omega)$ has its spectral image in l_2 according to bases chosen in each spaces. If one identifies the square of norm of $F(x) \in L_2(\Omega)$ with the energy or information it carries, in terminology of physics, the outcome of Theorems presented in this paper is to assert that for any given F there exists an optimal basis in $L_2(\Omega)$ which furnishes the element with an image sharply concentrated on a few of spectrum-lines in l_2, if the latter is equipped with canonical basis. This may be in marked contrast with a flat spread of spectral lines with respect to a casually chosen basis for spectral analysis as it happens in many cases of practices.

The results presented in this paper may find wide applications in computational mathematics and engineering sciences, particularly in the field of control theory and automation [20]. Take a typical example, if a hypersurface or mainfold in $\mathbb{R}^n$, $x^n = F(x^1, x^2, \ldots, x^{n-1})$, is needed to be stored, the amount of data is measured as $N^{n-1} + N, N$ is the mean number of discrete samplings for each variable. If l terms are taken in (13) to represent F, the amount of data to be stored or processed will be reduced to nlN, a $1/N^{n-2}$ times less than previously needed. Engineering practice had shown, sometimes to take two to three terms of (13) would be precise enough to represent a given higher dimensional function by the sum of products of one-variable functions [20].

The problems we investigated in this paper are related to a topic posed and studied by Liapunov A.M. at the beginning of 20^{th} century, he called it power series integral equations and imposed severe restriction on the given function. He required $F(x^1, x^2, \ldots, x^n)$ to be totally symmetric, that means the exchange of any two among n variables retains F unchanged [21,22]. The general properties of n-multilinear forms have been elucidated in [5, 6, 7]. Krasnoselsky M.A. proved that if F is strictly positive and totally symmetric, $F \in L_2(\Omega)$, $0 < m \leq F \leq M < \infty$, the following integral equation

$$\Phi\left(\prod_{i=1}^{n} \varphi(t_i)\right) = \\ = \int\limits_{\Omega} F(s, t_1, t_2, \cdots t_n \Big) \varphi(t_1)\varphi(t_2)\cdots\varphi(t_n) dt_1 dt_2 \cdots dt_n = \lambda\varphi(s)$$

possesses at least one positive eigenvalue [22]. Wainberg M.M. had shown that for a totally symmetric $F \in L_2(\Omega)$ all components of the gradient operator Φ generated by F are compact, and the functional $\left\langle \Phi\left(\prod_{i=1}^{n} \varphi(t_i)\right),\ \varphi(s) \right\rangle$ is weak continuous respect to φ, it achieves its supremum value on the unit ball [21]. It is obvious, the results we obtained in this paper cover most cases studied by these earlier investigators.

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