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REDUCTION OF BAND MATRICES IN LARGE DYNAMIC SYSTEMS CONTROLLABILITY AND OBSERVABILITY

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Abstract

An approach is proposed for linear stationary dynamical system with controllability and observability band matrices making it possible to simplify procedures for evaluating controllability and observability of this system. The obtained results are based on the fact that the controllability and observability criteria of a dynamic system are equivalent due to their required and sufficient properties; therefore, any transformations of one criterion not violating the conditions of necessity and sufficiency could be reduced to transformations in a sense equivalent to the initial transformations. The Popov — Belevich — Hautus transformations of the controllability and observability criteria were taken as a basis, and then results of such transformations were correctly extended to the band criteria. It was proved that the controllability and observability analysis of a linear stationary system with a large number of the state dimensions was reduced to studying the matrices rank of a much smaller size. The proposed approach is based on the existence condition for a numerical matrix of a certain rank of the nondegenerate matrices that satisfy certain transformations. The corresponding controllability and observability theorems for the stationary dynamical systems were provided. It was shown that for systems with one input and one output, the controllability and observability analysis was reduced to the analysis of scalars

Keywords

Linear stationary dynamical system, controllability and observability criteria, band matrices, matrix reduction

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Introduction. Let us consider a linear dynamical system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^r$ is the control vector; $y(t) \in \mathbb{R}^m$ is the output vector. It is assumed that $\operatorname{rank} B = r$, $\operatorname{rank} C = m$ and $n \gg 1$, i.e., the input and output matrices are having the corresponding full rank, and dimension of the system states (1) is a large number. Foreign literature introduces the concept of Large-Scale System for such system classes.

Various criteria are used for controllability and observability of the dynamical systems with representation in the state space (1). The most common are the Kalman criteria and the Popov — Belevich — Hautus tests [1–3]. According to these criteria, it is necessary and sufficient that the rank conditions for complete controllability and observability are fulfilled (1):

$$\operatorname{rank}\left[\begin{array}{c:c:c:c:c} B & AB & A^2B & \cdots & A^{n-r}B \end{array}\right] = n; \qquad \operatorname{rank}\left[\begin{array}{c} C \\ \hdashline CA \\ \hdashline CA^2 \\ \hdashline \vdots \\ \hdashline CA^{n-m} \end{array}\right] = m;$$

$$\forall \lambda \in \mathbb{C}: \quad \operatorname{rank}\left[\begin{array}{c:c} A - \lambda I_n & B \end{array}\right] = n; \tag{2}$$

$$\forall \lambda \in \mathbb{C}: \quad \operatorname{rank}\left[\begin{array}{c} A - \lambda I_n \\ \hdashline C \end{array}\right] = n. \tag{3}$$

Here I_n is the identity matrix of the size $n \times n$; $\mathbb{C}$ is the set of complex numbers (complex plane).

Let us note that the $\forall \lambda \in \mathbb{C}$ condition in (2), (3) could be replaced without losing generality by condition $\forall \lambda \in \Lambda(A)$, where $\Lambda(A) = \left\{ \lambda_i \middle| \det\left(A - \lambda_i I_n\right) = 0 \right\}$ is the set of the A matrix eigenvalues.

In the early 2000s, the criteria for controllability and observability of a dynamic system (1) were proposed, where controllability and observability were determined based on the rank examination [3, 4]:

– controllability band matrix sized $(n-r)(n-r+1) \times n(n-r)$:

$$\mathbf{C}_{band} = \left[\begin{array}{c:c:c:c:c} LA & 0 & 0 & \cdots & 0 \\ \hdashline L & LA & 0 & \cdots & 0 \\ \hdashline 0 & L & LA & \cdots & 0 \\ \hdashline 0 & 0 & L & \ddots & \vdots \\ \hdashline \vdots & \vdots & \vdots & \ddots & LA \\ \hdashline 0 & 0 & 0 & \cdots & L \end{array}\right]; \tag{4}$$

– observability band matrix sized $n(n-m)\times(n-m)(n-m+1)$:

$$\mathbf{O}_{band}=\left[\begin{array}{c:c:c:c:c:c} AR & R & 0 & \cdots & 0 & 0 \\ \hdashline 0 & AR & R & \cdots & 0 & 0 \\ \hdashline 0 & 0 & AR & \ddots & \vdots & \vdots \\ \hdashline \vdots & \vdots & \vdots & \ddots & R & 0 \\ \hdashline 0 & 0 & 0 & \cdots & AR & R \end{array}\right]. \tag{5}$$

Here $L,\ R$ are the (non-unique) solution matrices of the maximum rank [2, 5, 6] for the homogeneous equations:

$$LB=0; \tag{6}$$

$$CR=0. \tag{7}$$

The L and R matrices are also called the left and right zero divisors of the maximum rank [3–5, 7].

Proved in [3, 4] that for complete controllability of a dynamical system (1), it is necessary and sufficient that

$$\text{rank }\mathbf{C}_{band}=n(n-r+1). \tag{8}$$

Dually, it is necessary and sufficient for complete observability (1) that

$$\text{rank }\mathbf{O}_{band}=n(n-m+1). \tag{9}$$

Practical problems in analyzing controllability and observability of large dynamical systems (see, for example, [8–11]) often face situations, where dimension of the control and output vectors is comparable with the dimension of the state space. In this case, standard controllability tests for a system (1) are the high-dimensional ill-conditioned problems [12].

Recursive algorithms were proposed for band controllability and observability criteria (8), (9) with matrices (4), (5) in [12] that could significantly reduce the computational costs in large dynamic systems.

This work objective is to construct reductions that would make it possible to make a judgment on controllability and observability of a large dynamical system (1) based on studying the rank of matrices of a much smaller size.

Methodological basis of the work. For any $D\in\mathbb{R}^{p\times k}$ numerical matrix of the s rank, there are nonsingular matrices [3, 4, 7]:

$$\left[\begin{array}{c} L \\ \hdashline E_L \end{array}\right],\quad \left[\begin{array}{c:c} E_R & R \end{array}\right],$$

satisfying the following transformation:

$$\left[\begin{array}{c} E_L \\ \hdashline L \end{array}\right] D \left[\begin{array}{c:c} E_R & R \end{array}\right] = \left[\begin{array}{c:c} I_s & 0_{s\times(k-s)} \\ \hdashline 0_{(p-s)\times s} & 0_{(p-s)\times(k-s)} \end{array}\right]. \qquad (10)$$

Here $0_{i\times j}$ is the zero matrix of size $i\times j$. Thus, transformation in the form expanded to blocks (10) decomposes into four relations:

$$E_L D\, E_R = I_s;$$
$$L\, D\, E_R = 0_{(p-s)\times s};$$
$$E_L D\, R = 0_{s\times(k-s)};$$
$$L\, D\, R = 0_{(p-s)\times(k-s)}.$$

For system (1), the $B\in\mathbb{R}^{n\times r}$ matrix of the r rank with transformation by type (10) using matrix

$$\left[\begin{array}{c} B^{+} \\ \hdashline L \end{array}\right] \qquad (11)$$

is brought to the following form:

$$\left[\begin{array}{c} I_r \\ \hdashline 0_{(n-r)\times r} \end{array}\right].$$

Here $B^{+}=E_L$ is the Moore — Penrose pseudoinverse matrix [13]; L is the maximum solution of the homogeneous equation (8).

Similarly, the $C\in\mathbb{R}^{m\times n}$ matrix of the m rank from (1) using matrix

$$\left[\begin{array}{c:c} C^{+} & R \end{array}\right]$$

is transformed to the form $\left[\begin{array}{c:c} I_m & 0_{m\times(n-m)} \end{array}\right]$, where R is the maximum solution to the homogeneous equation (7); $C^{+}=E_R$ is the Moore — Penrose pseudoinverse matrix [13].

The described controllability and observability criteria of the dynamical system (1) are *equivalent* due to required and sufficient properties [1, 4]. Therefore, any transformations of one criterion not violating the necessity and sufficiency conditions could be reduced to transformations that are in some sense equivalent to the original transformations. For example, if controllability of one system from the Popov — Belevich — Hautus criterion entails controllability of some other system, then this conclusion could be extended both to the Kalman criterion and to the band matrices criterion.

The present work takes as the basis the Popov — Belevich — Hautus transformations of the controllability and observability criteria [14]; results of such transformations are further correctly extended to the band criteria.

Matrix reduction in the controllability criteria. Using matrix (11), we transform the $\left[A-\lambda I_n \mid B\right]$ matrix beam from criterion (2):

$$\left[\begin{array}{c} B^{+} \\ \hline L \end{array}\right]\left[A-\lambda I_n \mid B\right]=\left[\begin{array}{c|c} B^{+}\left(A-\lambda I_n\right) & I_r \\ \hline L\left(A-\lambda I_n\right) & 0_{(n-r)\times r} \end{array}\right].$$

Due to the matrix (11) nonsingularity, the following is obtained [13]:

$$\operatorname{rank}\left[A-\lambda I_n \mid B\right]=\operatorname{rank}\left[\begin{array}{c|c} B^{+}\left(A-\lambda I_n\right) & I_r \\ \hline L\left(A-\lambda I_n\right) & 0_{(n-r)\times r} \end{array}\right]. \tag{12}$$

It follows from the structure of matrix (12) that the $\left[B^{+}\left(A-\lambda I_n\right) \mid I_r\right]$ submatrix has the r rank for any λ values. In this regard, in order to fulfill condition (2), it is necessary and sufficient that the rank of the $L\left(A-\lambda I_n\right)$ matrix beam satisfies the requirement [3, 4]:

$$\forall\lambda\in\mathbb{C}: \quad \operatorname{rank}\ L\left(A-\lambda I_n\right)=n-r.$$

Let us introduce into consideration the nonsingular matrix

$$\left[L^{+} \mid B\right], \tag{13}$$

satisfying the following identity:

$$L\left[L^{+} \mid B\right]=\left[I_{n-r} \mid 0_{(n-r)\times r}\right].$$

Using (13), let us transform the $L\left(A-\lambda I_n\right)$ submatrix from (12):

$$L\left(A-\lambda I_n\right)\left[L_0^{+} \mid B\right]=\left[LAL^{+}-\lambda I_{n-r} \mid LAB\right].$$

In this case, as in the previous one (12),

$$\operatorname{rank}\ L\left(A-\lambda I_n\right)=\operatorname{rank}\left[LAL^{+}-\lambda I_{n-r} \mid LAB\right]. \tag{14}$$

To simplify, let us introduce the following notations:

$$\begin{aligned} &A^{(1)}=LAL^{+}; \quad B^{(1)}=LAB; \\ &n^{(1)}=n-r; \quad r^{(1)}=\operatorname{rank} B^{(1)}. \end{aligned} \tag{15}$$

In this case, let us assume that $L^{(1)}$ is the maximum solution to the homogeneous equation

$$L^{(1)}B^{(1)}=0. \tag{16}$$

Comparing the right parts of expressions (2) and (14) and taking into consideration equivalence of the considered criteria, validity of the following lemma is obtained.

Lemma 1. *The dynamical system (1) is controllable, if and only if the following equivalent conditions are satisfied*:

$$\operatorname{rank}\left[B^{(1)} \;\vdots\; A^{(1)}B^{(1)} \;\vdots\; \left(A^{(1)}\right)^2 B^{(1)} \;\vdots\; \cdots \;\vdots\; \left(A^{(1)}\right)^{n^{(1)}-r^{(1)}} B^{(1)} \right] = n^{(1)}, \tag{17}$$

$$\forall \lambda \in \Lambda\left(A^{(1)}\right): \ \operatorname{rank}\left[A^{(1)} - \lambda I_{n^{(1)}} \;\vdots\; B^{(1)} \right] = n^{(1)}, \tag{18}$$

$$\mathbf{C}_{band}^{(1)} = \left[\begin{array}{c:c:c:c:c} L^{(1)}A^{(1)} & 0 & 0 & \cdots & 0 \\ \hdashline L^{(1)} & L^{(1)}A^{(1)} & 0 & \cdots & 0 \\ \hdashline 0 & L^{(1)} & L^{(1)}A^{(1)} & \cdots & 0 \\ \hdashline 0 & 0 & L^{(1)} & \ddots & \vdots \\ \hdashline \vdots & \vdots & \vdots & \ddots & L^{(1)}A^{(1)} \\ \hdashline 0 & 0 & 0 & \cdots & L^{(1)} \end{array}\right]_{(n^{(1)}-r^{(1)})(n^{(1)}-r^{(1)}+1)\times n^{(1)}(n^{(1)}-r^{(1)})}, \tag{19}$$

where $\Lambda\left(A^{(1)}\right) = \Lambda\left(LAL^{+}\right)$ *is the set of eigenvalues of matrix* $A^{(1)}$; $A^{(1)}$, $B^{(1)}$, $L^{(1)}$ *are the matrices satisfying (15), (16).*

The transformations carried out resulted in reduction in the matrices size under the considered criteria. Thus, if we assume that n = 100, r = 40, $r^{(1)}$ = = 30 in system (1), then in (17) there is a controllability matrix of the 60 × 120 size (initial size is 100 × 2400), in (18) — a matrix beam of the 60 × 100 size (initial size is 100 × 140), and in (19) — a band matrix of the 930 × 1800 size (initial size is 3660 × 6000). The most significant decrease in size occurred in the controllability band matrix (almost by 4 times).

Let us consider transformation of the form (10) with respect to matrix $B^{(1)}$:

$$\left[\begin{array}{c} E_L^{(1)} \\ \hdashline L^{(1)} \end{array}\right] B_1 \left[E_R^{(1)} \;\vdots\; R^{(1)} \right] = \left[\begin{array}{c:c} I_{r^{(1)}} & 0_{r^{(1)}\times(r-r^{(1)})} \\ \hdashline 0_{(n_1^{(1)}-r^{(1)})\times r^{(1)}} & 0_{(n^{(1)}-r^{(1)})\times(r-r^{(1)})} \end{array}\right]. \tag{20}$$

Applying transformation (20) to the $\left[A_1 - \lambda I_{n_1} \;\vdots\; B_1 \right]$ matrix and taking into account the right side of (20), the following is obtained:

$$\left[\begin{array}{c} E_L^{(1)} \\ \hdashline L^{(1)} \end{array}\right] \left[A^{(1)} - \lambda I_{n^{(1)}} \;\vdots\; B^{(1)} \right] \left[E_R^{(1)} \;\vdots\; R^{(1)} \right] =$$

$$= \left[\begin{array}{c:c:c:c} E_L^{(1)}\left(A^{(1)} - \lambda I_{n^{(1)}}\right)E_R^{(1)} & E_L^{(1)}\left(A^{(1)} - \lambda I_{n^{(1)}}\right)R^{(1)} & I_{r^{(1)}} & 0_{r^{(1)}\times(r-r^{(1)})} \\ \hdashline L^{(1)}\left(A^{(1)} - \lambda I_{n^{(1)}}\right)E_R^{(1)} & L^{(1)}\left(A^{(1)} - \lambda I_{n^{(1)}}\right)R^{(1)} & 0_{(n_1^{(1)}-r^{(1)})\times r^{(1)}} & 0_{(n^{(1)}-r^{(1)})\times(r-r^{(1)})} \end{array}\right]. \tag{21}$$

Results of the analysis (21) show that in this case the chain of rank conditions is valid:

$$\operatorname{rank}\left[\begin{array}{c:c:c:c} E_L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)E_R^{(1)} & E_L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)R^{(1)} & I_{r^{(1)}} & 0_{r^{(1)}\times\left(r-r^{(1)}\right)} \\ \hdashline L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)E_R^{(1)} & L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)R^{(1)} & 0_{\left(n_1^{(1)}-r^{(1)}\right)\times r^{(1)}} & 0_{\left(n^{(1)}-r^{(1)}\right)\times\left(r-r^{(1)}\right)} \end{array}\right]=$$

$$=r^{(1)}+\operatorname{rank}\left[\begin{array}{c:c} L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)E_R^{(1)} & L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)R^{(1)} \end{array}\right]=$$

$$=r^{(1)}+\operatorname{rank}\ L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)\left[\begin{array}{c:c} E_R^{(1)} & R^{(1)} \end{array}\right]=r^{(1)}+\operatorname{rank}\ L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right). \quad (22)$$

Structure of the $L^{(1)}\left(A^{(1)}-\lambda I_{n^{(1)}}\right)$ matrix beam from (22) exactly corresponds to the $L\left(A-\lambda I_n\right)$ matrix beam structure from (12). On the basis of its properties, Lemma 1 was formulated. This fact makes it possible to formulate another assertion.

Lemma 2. *Dynamical system (1) is controllable, if and only if the following equivalent conditions are satisfied:*

$$\operatorname{rank}\left[\begin{array}{c:c:c:c:c} B^{(2)} & A^{(2)}B^{(2)} & \left(A^{(2)}\right)^2 B^{(2)} & \cdots & \left(A^{(2)}\right)^{n^{(2)}-r^{(2)}} B^{(2)} \end{array}\right]=n^{(2)}, \quad (23)$$

$$\forall\lambda\in\Lambda\left(A^{(2)}\right):\ \operatorname{rank}\left[\begin{array}{c:c} A^{(2)}-\lambda I_{n^{(2)}} & B^{(2)} \end{array}\right]=n^{(2)},$$

$$\mathbf{C}_{band}^{(2)}=\left[\begin{array}{c:c:c:c:c} L^{(2)}A^{(2)} & 0 & 0 & \cdots & 0 \\ \hdashline L^{(2)} & L^{(2)}A^{(2)} & 0 & \cdots & 0 \\ \hdashline 0 & L^{(2)} & L^{(2)}A^{(2)} & \cdots & 0 \\ \hdashline 0 & 0 & L^{(2)} & \ddots & \vdots \\ \hdashline \vdots & \vdots & \vdots & \ddots & L^{(2)}A^{(2)} \\ \hdashline 0 & 0 & 0 & \cdots & L^{(2)} \end{array}\right]_{\left(n^{(2)}-r^{(2)}\right)\left(n^{(2)}-r^{(2)}+1\right)\times n^{(2)}\left(n^{(2)}-r^{(2)}\right)},$$

where $A^{(2)}=L^{(1)}AL^{(1)+}$; $B^{(2)}=L^{(1)}A^{(1)}B^{(1)}$; $n^{(2)}=n^{(1)}-r^{(1)}$; $r^{(2)}=\operatorname{rank}B^{(2)}$; $L^{(2)}$ *is the maximum rank solution to homogeneous equation* $L^{(2)}B^{(2)}=0$; $\Lambda\left(A^{(2)}\right)=\Lambda\left(L^{(1)}AL^{(1)+}\right)$ *is the set of eigenvalues of matrix* $A^{(2)}$; $L^{(1)+}$ *is the Moore — Penrose pseudoinverse matrix.*

Continuing the reasoning by induction, the following theorem is obtained, which validity was already proven.

Theorem 1. *Dynamical system (1) is controllable, if and only if the following equivalent conditions are satisfied:*

$$\operatorname{rank}\left[B^{(i)} \;\vdots\; A^{(i)}B^{(i)} \;\vdots\; \left(A^{(i)}\right)^2 B^{(i)} \;\vdots\; \cdots \;\vdots\; \left(A^{(i)}\right)^{n^{(i)}-r^{(i)}} B^{(i)} \right] = n^{(i)};$$

$$\forall \lambda \in \Lambda\left(A^{(i)}\right):\ \operatorname{rank}\left[A^{(i)} - \lambda I_{n^{(i)}} \;\vdots\; B^{(i)} \right] = n^{(i)};$$

$$\mathbf{C}_{band}^{(i)} = \begin{bmatrix} L^{(i)}A^{(i)} & 0 & 0 & \cdots & 0 \\ L^{(i)} & L^{(i)}A^{(i)} & 0 & \cdots & 0 \\ 0 & L^{(i)} & L^{(i)}A^{(i)} & \cdots & 0 \\ 0 & 0 & L^{(i)} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & L^{(i)}A^{(i)} \\ 0 & 0 & 0 & \cdots & L^{(i)} \end{bmatrix}_{(n^{(i)}-r^{(i)})(n^{(i)}-r^{(i)}+1)\times n^{(i)}(n^{(i)}-r^{(i)})},$$

where $i = 1, 2, 3, \ldots \le n - r$ *are the natural numbers;* $A^{(i)} = L^{(i-1)} A L^{(i-1)+}$; $B^{(i)} = L^{(i-1)} A^{(i-1)} B^{(i-1)}$; $n^{(i)} = n^{(i-1)} - r^{(i-1)}$; $r^{(i)} = \operatorname{rank} B^{(i)}$; $L^{(i)} \ne 0$ *is the solution of the maximum rank of the homogeneous equation* $L^{(i)}B^{(i)} = 0$; $\Lambda\left(A^{(i)}\right)$ *is the set of eigenvalues of matrix* $A^{(i)}$; $L^{(i)+}$ *is the Moore — Penrose pseudoinverse matrix. At* $i = 1$: $A^{(0)} = A$, $B^{(0)} = B$, $n^{(0)} = n$, $r^{(0)} = r$.

The corollary obviously follows from Theorem 1.

Corollary 1. *Dynamical system (1), where* $n > 1$, $u(t) \in \mathbb{R}^1$ *is a scalar, is controllable if and only if the pair of scalars is controllable:*

$$\left(A^{(n-1)},\ B^{(n-1)}\right) = \left(L^{(n-2)} A^{(n-2)} L^{(n-2)+},\ \ L^{(n-2)} A^{(n-2)} B^{(b-2)}\right), \quad (24)$$

i.e., in (24), the scalar $B^{(n-1)} = L^{(n-2)} A^{(n-2)} B^{(n-2)} \ne 0$.

Matrix reduction in the observability criteria. To solve the problem of analyzing the system observability (1), let us use matrix transformations (3) dualized to the matrix transformations (2) performed above. As a result, the following statement is obtained.

Theorem 2. *Dynamical system (1) is observable, if and only if the following equivalent conditions are satisfied:*

$$\operatorname{rank}\begin{bmatrix} C^{(i)} \\ C^{(i)}\widehat{A}^{(i)} \\ C^{(i)}\left(\widehat{A}^{(i)}\right)^2 \\ \vdots \\ C^{(i)}\left(\widehat{A}^{(i)}\right)^{n-m} \end{bmatrix} = m^{(i)};$$

$$\forall\lambda \in \Lambda\left(A^{(i)}\right): \ \operatorname{rank}\left[\begin{array}{c}\widehat{A}^{(i)} - \lambda I_{n^{(i)}} \\ \hdashline C^{(i)}\end{array}\right] = n^{(i)};$$

$$\mathbf{O}_{band}^{(i)} = \left[\begin{array}{c:c:c:c:c:c} \widehat{A}^{(i)}R^{(i)} & R^{(i)} & 0 & \cdots & 0 & 0 \\ \hdashline 0 & \widehat{A}^{(i)}R^{(i)} & R^{(i)} & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \widehat{A}^{(i)}R^{(i)} & \ddots & \vdots & \vdots \\ \hdashline \vdots & \vdots & \vdots & \ddots & R^{(i)} & 0 \\ \hdashline 0 & 0 & 0 & \cdots & \widehat{A}^{(i)}R^{(i)} & R^{(i)} \end{array}\right]_{(n^{(i)}-m^{(i)})(n^{(i)}-m^{(i)}+1)\times n^{(i)}(n^{(i)}-m^{(i)})},$$

where $i = 1, 2, 3, \ldots \le n - m$ *are the natural numbers;* $\widehat{A}^{(i)} = R^{(i-1)+}\widehat{A}R^{(i-1)}$; $B^{(i)} = C^{(i-1)}\widehat{A}^{(i-1)}R^{(i-1)}$; $n^{(i)} = n^{(i-1)} - m^{(i-1)}$; $m^{(i)} = \operatorname{rank} C^{(i)}$; $R^{(i)} \ne 0$ *is the solution of the maximum rank of the homogeneous equation* $C^{(i)}R^{(i)} = 0$; $\Lambda\left(\widehat{A}^{(i)}\right)$ *is the set of eigenvalues of matrix* $\widehat{A}^{(i)}$; $R^{(i)+}$ *is the Moore — Penrose pseudoinverse matrix. At* $i = 1$:

$$\widehat{A}^{(0)} = A, \quad C^{(0)} = C, \quad n^{(0)} = n, \quad m^{(0)} = m.$$

Corollary 2. *System (1), where* $n > 1$, $y(t) \in \mathbb{R}^1$, is the scalar and is observable, if and only if the pair of scalars is observable:

$$\left(\widehat{A}^{(n-1)}, \ C^{(n-1)}\right) = \left(R^{(n-2)+}\widehat{A}^{(n-2)}R^{(n-2)}, \quad C^{(n-2)}\widehat{A}^{(n-2)}R^{(n-2)}\right),$$

i.e., in (36) the scalar $C^{(n-1)} = C^{(n-2)}\widehat{A}^{(n-2)}R^{(n-2)} \ne 0.$

Conclusion. It was proved that analysis of controllability and observability of a dynamical system with a large number of state dimensions by reduction lies in studying the rank of matrices of much smaller size. For systems with one input and one output, the controllability and observability analysis are reduced to analyzing the scalars. The result obtained is of great importance and simplifies the analysis and synthesis procedure in control systems and the solution of the above mathematical problems in conjunction with other areas of research by the authors on the modal control methods [15–18] and the possibilities of its application, in particular, solving the algebraic equations [19].

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