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DRY FRICTION AND MECHANICAL SYSTEM MOTION IMPLICIT EQUATIONS

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Abstract

It is shown that forces acting on the mechanical system points could depend on accelerations of the system points. Differential equation system of the mechanical system motion appears to be implicit. It is not resolved with respect to senior derivatives. Fundamental mathematical problems appear associated with possibility and uniqueness of these equations' solution with respect to the senior derivatives. Such problems are common in mechanical systems with dry sliding friction and rolling friction. Such problems are missing in the point dynamics. However, such problems are rather typical in more complex mechanical systems appearing in the study of a rigid body motion, which entire mass is concentrated in a single point, as well as in systems with one degree of freedom. Four fairly simple examples of mechanical systems are considered, and their motion is described by implicit differential motion equations. Situations could appear in these systems, when motion equations are not solvable with respect to the senior derivatives (motion equations are missing), as well as situations, when there are several solutions with respect to senior derivatives (there are several different systems of the mechanical system motion equations). At the same time, one of the fundamental principles of mechanics is not fulfilled, i.e., the principle of determinism

Keywords

Nonlinear dynamics, dry friction, implicit differential motion equations, Painlevé paradoxes

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Introduction. Axioms of dynamics are based on the determinism principle [1], which states that initial state of a mechanical system unambiguously determines further system behavior exposed to the given forces action. Determinism principle is a particular case of the principle of experience repeatability in physics. If one and the same experiment is carried out under the same conditions, one and the same result is obtained.

Determinism principle served as the basis in formulation of Newton's second law, according to which differential equation of the material point motion exposed to the $\overline{F}$ force action has the following form:

$$m\ddot{\overline{r}} = \overline{F}\left(\overline{r}, \overline{v}, t\right), \tag{1}$$

where m is the point's mass; $\overline{r}$ is its radius vector; $\overline{v} = \dot{\overline{r}}$ is the point velocity; $\overline{F}$ is the force being the function of the point position, velocity and time. If the initial state of the point is set as:

$$\overline{r}\left(t_0\right) = \overline{r}_0, \;\; \dot{\overline{r}}\left(t_0\right) = \overline{v}_0, \tag{2}$$

then the main problem of dynamics in determining further motion of the point is the Cauchy problem. And it has a unique solution, if conditions of the theorem of the differential equations' existence and unique solution are satisfied [1–5].

Note that although mechanics assumes by default that all mechanical systems are deterministic, there are also non-deterministic systems. Examples of such systems for the simplest case of the material point rectilinear motion are provided in [4–6]. In these systems, the one and the same initial state (2) may correspond to several different solutions of the motion equations (1). Examples of mechanical system nondeterministic behavior in the impact theory are considered in [7].

Problem statement. Differential equation of the point motion (1) is transferred to the mechanical system. For the n material points system, the motion differential equations have the following form [1–5]:

$$m_k \ddot{\overline{r}}_k = \overline{F}_k, \;\; k = 1, 2, \ldots, n, \tag{3}$$

where m_k is the mass of the system k-th point; $\overline{r}_k$ is its radius vector. Force $\overline{F}_k$ acting on the system k-th point is a function of position and velocities of all points in the system and time:

$$\overline{F}_k = \overline{F}_k\left(\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_n, \dot{\overline{r}}_1, \dot{\overline{r}}_2, \ldots, \dot{\overline{r}}_n, t\right), \;\; k = 1, 2, \ldots, n.$$

Consequently, equations of the mechanical system motion form a system of differential equations resolved with respect to the senior derivatives.

However, situations are possible, when forces acting on the system points also depend on the system points' accelerations:

$$\overline{F}_k = \overline{F}_k\left(\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_n, \dot{\overline{r}}_1, \dot{\overline{r}}_2, \ldots, \dot{\overline{r}}_n, \ddot{\overline{r}}_1, \ddot{\overline{r}}_2, \ldots, \ddot{\overline{r}}_n, t\right), \;\; k = 1, 2, \ldots, n. \tag{4}$$

The system of differential equations of the mechanical system motion appears to be implicit. It is not solved with respect to the senior derivatives. Fundamental mathematical problems appear associated with possibility and unique solution of these equations with respect to the senior derivatives.

Naturally, implicit form of the motion equations is preserved when passing to generalized coordinates using general dynamics theorems or Lagrange equations of the second kind.

Such situations are typical for mechanical systems with dry friction [8–11]. Such situations do not appear in the point dynamics, but it could become rather typical in more complex mechanical systems, including cases of studying motion of a rigid body, which mass is concentrated in a single point, as well as in systems with one degree of freedom.

This is due to the fact that in accordance with Coulomb's law the dry friction force in sliding has the following form:

$$\overline{F}_{fr} = -f\left|\overline{N}\right|\frac{\overline{v}}{\left|\overline{v}\right|}, \tag{5}$$

where f is the sliding friction coefficient; $\overline{N}$ is the normal reaction; $\overline{v}$ is the relative sliding speed. Normal reaction could depend on the system points accelerations. Then, motion equations have the implicit form (3), (4). In problems with dry friction, another complication arises due to the fact that at $v = 0$ the dry friction force could take any value in modulus not exceeding the maximum $\left|\overline{F}_{fr}\right| \le f\left|\overline{N}\right|$, and is directed in the direction opposite to the direction of possible sliding. This could lead to the existing stagnation areas, and falling there at zero speed leads to cessation of sliding.

If the bodies are able not only to slide relative to each other, but also roll, then, in addition to the sliding friction force, the rolling friction moment arises determined by similar ratios. Alternations of sliding phases, of rolling with slipping and rolling without slipping become possible.

Such systems were the subject of discussion in connection with the Painlevé paradoxes [8–16] associated with the fact that motion equations in certain cases turn out to be either unsolvable with respect to senior derivatives, or have several solutions. In other words, either there are no differential system motion equations, or there are several different systems of the system motion equations. This is contrary to the determinism principle.

As noted above, nondeterministic behavior of a mechanical system is also possible in systems without friction with motion equations resolved with respect to the senior derivatives.

Works [17, 18] are devoted to mathematical conditions for existence and unique solution of implicit differential equations of the mechanical systems motion in the general form.

Let us dwell on some examples of implicit equations of the mechanical systems motion with dry sliding friction and rolling friction. Let us consider a slightly modified Painlevé example and three more examples of mechanical systems with friction, which lead to implicit differential equations of the system motion. In all examples, situations arise similar to the Painlevé paradoxes.

Homogeneous bar (ladder). The bar restson horizontal floor and leans on vertical wall (Fig. 1). Bar mass is m, bar length is $AB = 2l$. Point C is the bar center of mass, $AC = CB = l$. Sliding friction coefficients at points A and B are equal to f. Bar position is determined by angle φ, which it forms with the vertical.

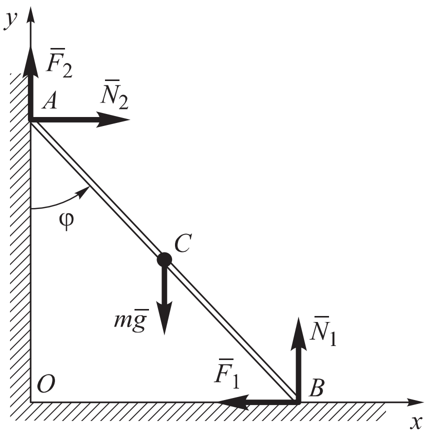

Fig. 1. Homogeneous bar (ladder)

At the $t = 0$ initial moment of time, the speed of point A is directed downward:

$$\varphi(0) = \varphi_0, \ \dot{\varphi}(0) = \omega_0 > 0. \qquad (6)$$

Let us denote by x, y coordinates of the C bar center of mass, then

$$x = l\sin\varphi, \quad y = l\cos\varphi.$$

Differentiating these constraint equations twice, we obtain

$$\ddot{x} = l\cos\varphi\,\ddot{\varphi} - l\sin\varphi\dot{\varphi}^2, \quad \ddot{y} = -l\sin\varphi\,\ddot{\varphi} - l\cos\varphi\dot{\varphi}^2. \qquad (7)$$

Let us denote by $\bar{N}_1$, $\bar{N}_2$ the normal reactions, and by $\bar{F}_1$, $\bar{F}_2$ the friction forces (see Fig. 1). Let us consider the stage of the bar motion, while contact with the floor and the wall is maintained, i.e., $N_1 \ge 0$ and $N_2 \ge 0$. In accordance with Coulomb's law (5):

$$F_1 = fN_1, \ F_2 = fN_2.$$

The theorem on the center of mass motion has the following form:

$$m\ddot{x} = N_2 - fN_1, \ m\ddot{y} = -mg + N_1 + fN_2.$$

Resolving these relations with respect to $\bar{N}_1$, $\bar{N}_2$, the following is obtained:

$$N_1 = \frac{-f\ddot{x} + g + \ddot{y}}{1+f^2}, \ N_2 = \frac{\ddot{x} + f\left(g + \ddot{y}\right)}{1+f^2}.$$

In accordance with the theorem on alteration in the angular momentum relative to the center of mass:

$$\frac{ml^2}{3}\ddot{\varphi} = \left(N_1 - fN_2\right) l \sin\varphi - \left(N_2 + fN_1\right) l \cos\varphi.$$

From the last two relations and (7), the implicit differential motion equation is obtained:

$$\left(1+f^2\right)\ddot{\varphi} = 3\left[2f\dot{\varphi}^2 - \left(1-f^2\right)\ddot{\varphi} - 2fg\cos\varphi + \left(1-f^2\right)g\sin\varphi\right],$$

which is easily resolved with respect to the senior derivative:

$$2\left(2-f^2\right)\ddot{\varphi} = 3\left[2f\dot{\varphi}^2 - 2fg\cos\kappa + \left(1-f^2\right)g\sin\varphi\right]. \qquad (8)$$

At $f = \sqrt{2}$, the motion equation does not contain acceleration. It has no solutions for acceleration, if the initial state does not satisfy the condition of equality to zero of the right-hand side of equation (8). Differential system motion equation does not exist. An analogue of the Painlevé paradoxes is obtained.

Differential motion equation (8) at $f \neq \sqrt{2}$ has the first integral. Let us denote by $\omega = \dot{\varphi}$ the angular velocity, and using substitution

$$\ddot{\varphi} = \frac{d\omega}{dt} = \frac{d\omega}{d\varphi}\frac{d\varphi}{dt} = \omega\frac{d\omega}{d\varphi} = \frac{1}{2}\frac{d\omega^2}{d\varphi} \qquad (9)$$

let us exclude time from equation (8). Then, an inhomogeneous linear differential equation is obtained with constant coefficients for dependence of the ω^2 angular velocity square on the rotation angle φ:

$$\frac{2-f^2}{3}\frac{d\omega^2}{d\varphi} - 2f\omega^2 = -2fg\cos\varphi + \left(1-f^2\right)g\sin\varphi,$$

which has the following solution: $\omega^2 = ce^{\lambda\varphi} + a\sin\varphi + b\cos\varphi,$ where

$$a = \frac{-6fg\left(5-4f^2\right)}{4+32f^2+f^4}; \quad b = \frac{-3g\left(2-7f^2+f^4\right)}{4+32f^2+f^4};$$

$$c = \left(\omega_0^2 - a\sin\varphi_0 - b\cos\varphi_0\right)e^{-\lambda\varphi_0}; \quad \lambda = \frac{6f}{2-f^2}.$$

Wheel with displaced center of mass. Inhomogeneous disc is rolling without slipping along a horizontal guide (Fig. 2). The disc is in the vertical plane. The C disc center of mass does not coincide with its geometric center A. Mass of the disc is m, disc radius is r, distance from the disc center to the center of mass is $AC = l$, disc moment of inertia relative to the center of mass is $J_c = m\rho^2$. Disc position is determined by generalized coordinate φ. The rolling friction coefficient equals to δ. The disc is rolling by inertia and is not bouncing over the supporting surface.

Let us denote by x, y coordinates of the wheel center point A, through x_c, y_c — coordinates of the disc center of mass. Then

$$x_c = x - l\sin\varphi = r\varphi - l\sin\varphi;$$
$$y_c = y - l\cos\varphi = r - l\cos\varphi.$$

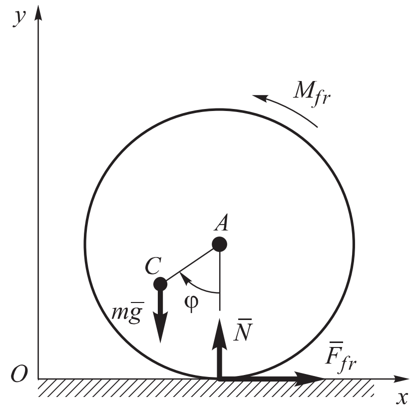

Fig. 2. Wheel with displaced center of mass

Differentiating these relations twice, the following is obtained:

$$\ddot{x}_c = r\ddot{\varphi} - l\cos\varphi\ddot{\varphi} + l\sin\varphi\dot{\varphi}^2; \;\; \ddot{y}_c = l\sin\varphi\ddot{\varphi} + l\cos\varphi\dot{\varphi}^2. \qquad (10)$$

In accordance with the theorem on the center of mass motion and the theorem on the angular momentum alteration relative to the center of mass, the following is obtained:

$$m\ddot{x}_c = -F_{fr}, \;\; m\ddot{y}_c = N - mg, \;\; m\rho^2\ddot{\varphi} = -M_{fr} + F_{fr}\left(r - l\cos\varphi\right) - Nl\sin\varphi. \qquad (11)$$

Then, taking into account (10), the following is obtained from the first two equations (11):

$$F_{fr} = -m\left(r\ddot{\varphi} - l\cos\varphi\ddot{\varphi} + l\sin\varphi\dot{\varphi}^2\right); \; N = mg + ml\left(\sin\varphi\ddot{\varphi} + \cos\varphi\dot{\varphi}^2\right).$$

Substituting these relations into the third of equations (11), let us write

$$m\left[\left(\rho^2 + r^2 + l^2 - 2rl\cos\varphi\right)\ddot{\varphi} + rl\sin\varphi\dot{\varphi}^2\right] = -mgl\sin\varphi - M_{fr}.$$

Suppose that the disk is not bouncing over the supporting surface, i.e., $N \geq 0$. For this, it is necessary and sufficient that $\dot{\varphi}$ and $\ddot{\varphi}$ do not exceed the critical values:

$$\min_t\left(\sin\varphi\ddot{\varphi} + \cos\varphi\dot{\varphi}^2\right) = \min_t \frac{d\left(\sin\varphi\dot{\varphi}\right)}{dt} \geq -\frac{g}{l}. \qquad (12)$$

The rolling friction moment is

$$M_{fr} = \delta N\,\mathrm{sgn}\,\dot{\varphi} = \delta m\left[g + l\left(\sin\varphi\ddot{\varphi} + \cos\varphi\dot{\varphi}^2\right)\right]\mathrm{sgn}\,\dot{\varphi}.$$

Implicit differential equation of the wheel motion has the following form:

$$m\left[\left(\rho^2 + r^2 + l^2 - 2rl\cos\varphi\right)\ddot{\varphi} + rl\sin\varphi\dot{\varphi}^2\right] =$$
$$= -mgl\sin\varphi - \delta mg\,\mathrm{sgn}\,\varphi - \delta m\left(l\sin\varphi\ddot{\varphi} + l\cos\varphi\dot{\varphi}^2\right)\mathrm{sgn}\,\dot{\varphi}$$

and is easily resolved with respect to the senior derivative. As a result

$$\left(\rho^2 + r^2 + l^2 - 2rl\cos\varphi + \delta l \sin\varphi \operatorname{sgn}\dot{\varphi}\right)\ddot{\varphi} =$$

$$= l\left(r\sin\varphi + \delta\cos\varphi \operatorname{sgn}\dot{\varphi}\right)\dot{\varphi}^2 - gl\sin\varphi - \delta g \operatorname{sgn}\dot{\varphi}. \qquad (13)$$

Equation (13) has no solutions for acceleration, when the coefficient in front of $\ddot{\varphi}$ is zero. The Painlevé paradoxes analogue is obtained.

Next, let us consider the case, where coefficient in front of $\ddot{\varphi}$ is not equal to zero.

Note that it is necessary to substitute $\ddot{\varphi}$ from equation (13) into condition (12) of the disk non-bouncing over the supporting surface. Then, it has the following form:

$$\min_t\left[-\frac{l\left(r\sin\varphi + \delta\cos\varphi \operatorname{sgn}\dot{\varphi}\right)\dot{\varphi}^2 + gl\sin\varphi + \delta g \operatorname{sgn}\dot{\varphi}}{\rho^2 + r^2 + l^2 - 2rl\cos\varphi + \delta l\sin\varphi \operatorname{sgn}\dot{\varphi}}\sin\varphi + \cos\varphi\,\dot{\varphi}^2\right] \geq -\frac{g}{l}.$$

According to definition of the function sgn:

$$\operatorname{sgn} 0 \in [-1, 1], \qquad (14)$$

i.e., it is able to take any value according to a module not exceeding 1.

From (14) it follows that the motion equation (13) has the stagnation zones. If the disk falls with the $\dot{\varphi} = 0$ zero angular velocity into the region, where

$$\left|\sin\varphi\right| \leq \frac{\delta}{l}, \qquad (15)$$

then it follows from (14) that $\ddot{\varphi} = 0,$ therefore, the disk stops.

If friction is high $\delta \geq l,$ then stagnation zone (15) covers all possible disc positions. The disc moves in one direction (without changing the angular velocity sign).

If the friction is low $\delta < l,$ then stagnation zones (15) form symmetric regions in the positions' vicinity, when the center of mass stays on the same vertical line with the disc geometric center, i.e., the extreme upper and extreme lower positions of the center of mass. If zero value of the angular velocity is reached during the disc first movement outside the stagnation zone, then disc motion at the final stage acquires the form of damped oscillations and ends with a stop in the stagnation zone near the lowest position of the center of mass.

Using substitution (9), the motion equation (13) is reduced to an inhomogeneous linear differential equation with variable coefficients for dependence of the ω^2 angular velocity square on the angle of rotation φ:

$$\left(\rho^2 + r^2 + l^2 - 2rl\cos\varphi + \delta l\sin\varphi\,\mathrm{sgn}\,\omega\right)\frac{d\omega^2}{d\varphi} +$$
$$+2l\left(r\sin\varphi + \delta\cos\varphi\,\mathrm{sgn}\,\omega\right)\omega^2 = -2gl\sin\varphi - 2\delta g\,\mathrm{sgn}\,\omega,$$

which determines the system phase trajectories.

Painlevé paradoxes. Elliptical pendulum is moving in the vertical plane (Fig. 3). The A slider moves along a rough horizontal guide. The sliding friction coefficient is equal to f. The AB weightless bar with the length of $2l$, at which end there is the B material point, is pivotally attached to the slider. Slider dimensions could be neglected. Masses of the A slider and the B material point are the same and equal to m. Friction in the A hinge could be neglected. Let us denote by $\overline{N}$ and $\overline{F}_{fr}$ both the normal reaction and the sliding friction force acting on the A slider.

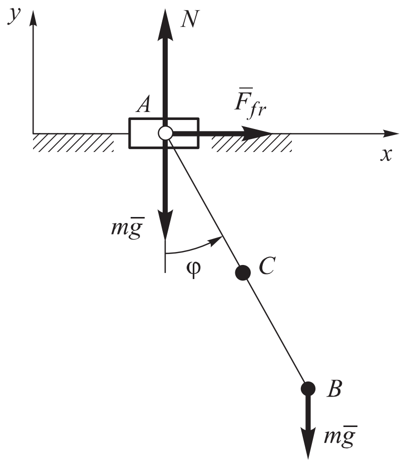

Fig. 3. Elliptical pendulum

Let us introduce the generalized coordinates: x is the A slider position; φ is the angle of bar deviation from the vertical. The C system center of mass is located in the $AC = CB = l$ bar middle, and its coordinates are equal to $x_c = x + l\sin\varphi,\ \ y_c = -l\cos\varphi$. Twice differentiating these ratios, the following is obtained:

$$\ddot{x}_c = \ddot{x} + l\cos\varphi\ddot{\varphi} - l\sin\varphi\dot{\varphi}^2,\ \ \ddot{y}_c = l\sin\varphi\ddot{\varphi} + l\cos\varphi\dot{\varphi}^2.$$

In accordance with the theorem on the center of mass motion, we have

$$2m\ddot{x}_c = 2m\left(\ddot{x} + l\cos\varphi\ddot{\varphi} - l\sin\varphi\dot{\varphi}^2\right) = F_{fr}; \tag{16}$$

$$2m\ddot{y}_c = 2ml\left(\sin\varphi\ddot{\varphi} + \cos\varphi\dot{\varphi}^2\right) = N - 2mg. \tag{17}$$

It follows from the theorem on alteration in the angular momentum with respect to the center of mass that

$$2ml^2\ddot{\varphi} = -Nl\sin\varphi - F_{fr}l\cos\varphi. \tag{18}$$

Substituting (16) and (17) into (18), the following is obtained:

$$2l\ddot{\varphi} + \ddot{x}\cos\varphi + g\sin\varphi = 0. \tag{19}$$

By virtue of the Coulomb's law (5):

$$F_{fr} = -f\left|N\right|\mathrm{sgn}\,\dot{x} = -kfN, \tag{20}$$

where

$$k = \operatorname{sgn}\left(N\dot{x}\right). \tag{21}$$

At $N\dot{x} \neq 0,\ k = \pm 1,$
and

$$kN\dot{x} > 0. \tag{22}$$

Let us substitute (20) into (16) and solve (16), (17), and (19) with respect to $N,\ \ddot{x},\ \ddot{\varphi}$ as a system of three linear algebraic equations. As a result

$$\lambda N = 2\left(g + l\cos\varphi\dot{\varphi}^2\right); \tag{23}$$

$$\lambda\ddot{x} = 2l\dot{\varphi}^2\left(\sin\varphi - kf\cos\varphi\right) - kfg\left(1+\cos^2\varphi\right)g\sin\varphi + g\sin\varphi\cos\varphi; \tag{24}$$

$$\lambda l\ddot{\varphi} = \left(g + l\cos\varphi\dot{\varphi}^2\right)\left(kf\cos\varphi - \sin\varphi\right). \tag{25}$$

Here

$$\lambda = 1 + \sin^2\varphi - kf\sin\varphi\cos\varphi. \tag{26}$$

Equations (24), (25) form a system of the motion differential equations resolved with respect to the senior derivatives. Equations (21)–(23) and (26) make it possible to determine the *k* and λ values. In this case, λ is uniquely determined by the *k* value. The question remains about the unambiguousness of their solution with respect to *k*.

Let the *f* friction coefficient be low enough, so that for any values φ:

$$1 + \sin^2\varphi - f\sin\varphi\cos\varphi > 0 \text{ and } 1 + \sin^2\varphi + f\sin\varphi\cos\varphi > 0. \tag{27}$$

Then λ is always higher than zero, and in any current state the $x,\ \varphi,\ \dot{x},\ \dot{\varphi}$ relations (21)–(23) and (26) uniquely determine the *k* and λ values.

Note that conditions (27) are satisfied for $f \le 2.$ Indeed, in this case

$$1 + \sin^2\varphi \pm f\sin\varphi\cos\varphi \ge 1 + \sin^2\varphi - f\left|\sin\varphi\cos\varphi\right| > 1 +$$
$$+ \sin^2\varphi - 2\left|\sin\varphi\cos\varphi\right| = 1 + \left(\left|\sin\varphi\right| - \left|\cos\varphi\right|\right)^2 - \cos^2\varphi > 1 - \cos^2\varphi \ge 0.$$

Let the friction coefficient be high enough and condition (27) is not satisfied. For definiteness, let us assume that in the current system state $\varphi \in \left(0,\ (1/2)\pi\right)$, then

$$1 + \sin^2\varphi - f\sin\varphi\cos\varphi < 0 \text{ and } 1 + \sin^2\varphi + f\sin\varphi\cos\varphi > 0. \tag{28}$$

Hence, it follows that for $\dot{x} > 0$ it is impossible to satisfy conditions (21) (23), (26). For $k = 1,$ it follows from (26), (28) that $\lambda < 0,$ and from (23) — $N < 0,$ then condition (22) is not satisfied. For $k = -1,$ we have $\lambda > 0$ and $N > 0,$ then condition (22) is not satisfied. Motion equations are undecidable with respect to the senior derivatives; no solution of the form (24), (25) exists.

Under $\dot{x} < 0$, conditions (21)–(26) are satisfied by both $k = \pm 1$ solutions, and there are two different types of the motion equations (24), (25). Consequently, the determinism principle is not fulfilled.

These contradictions to the basic principles of mechanics published by P. Painlevé [8] are called the Painlevé paradoxes [9–16].

It also should be noted that degeneration is observed at $\lambda = 0$. In this case, equations (22)–(23) have no solutions for accelerations. Another paradox that Painlevé did not mention.

Mathematical pendulum (Fig. 4) rotates around the horizontal rotation axis, i.e., it moves in a vertical plane around the O hinge. The pendulum consists of a weightless bar with a length of $OA = l$. At the end of the bar there is the A material point with the m mass. Pendulum position is determined by its φ angle of the bar deviation from the vertical.

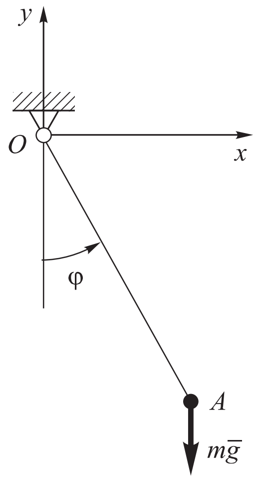

Fig. 4. Pendulum

Suppose that dry friction is in O the hinge, and it is reduced to the friction moment, which is proportional to the R reaction modulus in the O hinge:

$$M_{fr} = -\delta |R| \operatorname{sgn} \dot{\varphi}, \qquad (29)$$

δ is the friction coefficient of friction having the length dimension.

By virtue of the theorem on alteration in the angular momentum with respect to the O point, we have

$$ml^2 \ddot{\varphi} = -mgl \sin \varphi + M_{fr}. \qquad (30)$$

From the theorem on the center of mass motion it follows that in projection on the natural trihedron axis:

$$ma_n = ml\dot{\varphi}^2 = -mg \cos \varphi + R;\ ma_\tau = ml\ddot{\varphi} = -mgl \sin \varphi + R_\tau.$$

Then

$$|R| = \sqrt{R_n^2 + R_\tau^2} = m\sqrt{\left(l\dot{\varphi}^2 + g \cos \varphi\right)^2 + \left(l\ddot{\varphi} + g \sin \varphi\right)^2}. \qquad (31)$$

It follows from (29)–(31) that differential equation of the pendulum motion in implicit form is an irrational equation with respect to the senior derivative and has the following form:

$$l\left(l\ddot{\varphi} + g \sin \varphi\right) = -\delta\sqrt{\left(l\dot{\varphi}^2 + g \cos \varphi\right)^2 + \left(l\ddot{\varphi} + g \sin \varphi\right)^2} \operatorname{sgn} \varphi. \qquad (32)$$

Getting rid of irrationality by squaring, the following is obtained:

$$\left(l^2-\delta^2\right)\left(l\ddot{\varphi}+g\sin\varphi\right)^2=\delta^2\left(l\dot{\varphi}^2+g\cos\varphi\right)^2. \tag{33}$$

For large values of the friction coefficient $\delta \geq l,$ equation (33) has no solutions. We are facing a paradox similar to the Painlevé paradoxes. Differential equation of the mechanical system motion is missing. Consequently, the system motion is impossible.

For $\delta < l,$ equation (33) has two solutions:

$$l\ddot{\varphi}=-g\sin\varphi\pm\frac{\delta}{\sqrt{l^2-\delta^2}}\left|l\dot{\varphi}^2+g\cos\varphi\right|.$$

One of them is extraneous obtained by squaring equation (32). By virtue of (32), only one of these solutions is suitable:

$$l\ddot{\varphi}=-g\sin\varphi-\frac{\delta}{\sqrt{l^2-\delta^2}}\left|l\dot{\varphi}^2+g\cos\varphi\right|\operatorname{sgn}\dot{\varphi}, \tag{34}$$

which is the explicit differential equation of the pendulum motion.

Equation (24) has stagnation zones. If the pendulum falls with zero angular velocity $\dot{\varphi}=0$ into the region, where

$$\left|\operatorname{tg}\varphi\right|\leq\frac{\delta}{\sqrt{l^2-\delta^2}}, \tag{35}$$

then from (14) it follows that the right-hand side in the motion equation (34) is equal to zero, and the pendulum stops. Condition (35) is satisfied in the vicinity of the extreme lower and the extreme upper positions of the pendulum.

For a pendulum in weightlessness $(g=0),$ the motion differential equation (34) takes the following form:

$$\ddot{\varphi}=-\frac{\delta}{\sqrt{l^2-\delta^2}}\dot{\varphi}^2\operatorname{sgn}\dot{\varphi}. \tag{36}$$

Dry friction manifests itself as viscous (in this case, proportional to the angular velocity square).

Equation (36) could be easily solved analytically. Direction of rotation is not changing. For definiteness, let us assume that $\varphi(0)=\varphi_0,$ $\omega(0)=\omega_0>0,$ then

$$\omega=\frac{\omega_0}{1+\omega_0 kt},\quad \varphi=\varphi_0+\frac{1}{k}\ln\left(1+\omega_0 kt\right),$$

where

$$k=\frac{\delta}{\sqrt{l^2-\delta^2}}.$$

Conclusion. Situations are rather typical in mechanical systems with dry friction, when motion of a system is described by implicit differential motion equations unresolved with respect to the senior derivatives. These equations may turn out to be unsolved with respect to the senior derivatives or have several solutions. Such situations are called the Painlevé paradoxes.

In all the examples considered, dry friction leads to the appearance of forces (moments) of resistance proportional to the square of velocity after resolving the implicit motion equations with respect to the senior derivatives, i.e., dry friction also manifests itself as viscous. A similar effect appears in systems with transformed dry friction and in the multicomponent dry friction models [13, 14].

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